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ANALYTICAL STATICS.

Physics
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308

A TREATISE ON

ANALYTICAL STATICS

WITH NUMEROUS EXAMPLES.

By I. TODHUNTER, M.A., F.R.S.

FIFTH EDITION

EDITED BY

J. D. EVERETT, M.A., F.R.S.

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PREFACE TO THE FOURTH EDITION.

In this work will be found all the Propositions which usually appear in treatises on Theoretical Statics. To the different Chapters Examples are appended, which have been principally selected from the University and College Examination Papers; these will furnish ample exercise in the application of the principles of the subject.

Some of the Examples in the earlier Chapters assume results which are obtained at a later part of the book; the student who has no previous acquaintance with the subject may therefore, on his first perusal of the book, omit the more difficult Examples of the first six Chapters.

The subject besides being valuable for its own sake is important as the first, according to the usual order of study, which involves the application of the higher parts of pure mathematics to physics. Thus, for instance, the Chapters on the Centre of Gravity and on Attraction, besides their direct use in establishing interesting results in Mechanics, are indirectly of great advantage by illustrating the processes of the Integral Calculus, and special attention

has been paid to this circumstance by the discussion of numerous examples.

In the fourth edition the work has been revised, and some additions have been made which it is hoped will increase its utility.

I. TODHUNTER.

St John's College, Cambridge
June, 1874.

EDITOR'S PREFACE.

FINDING that the fourth edition of this work was out of print, and knowing of no other suitable text-book of Statics for my own honour class, I have, with the consent of the representatives of the late Dr Todhunter, undertaken the preparation of a new edition, with some changes intended to adapt it more closely to present requirements. The most important changes in the old matter relate to Attraction, Virtual Velocities, and General Theorems on Systems of Forces. The additional matter includes a brief chapter on Graphical Statics, a series of articles on the connection between Centres of Gravity and Resultants of Forces at a point (with an exposition of Vectors), and a new theorem on a String under a Central Force.

The book in its present form is intended to contain such a selection of subjects as may with advantage be studied in a First Course of Analytical Statics; and in conformity with this design I have, after considerable hesitation, omitted most of the articles on the Attraction of Ellipsoids, as well as Arts. 275, 276 of the supplementary Chapter XV. which was added in the fourth edition. The remainder of this supplementary chapter has been substantially incorporated.

The following is a nearly complete list of new and changed Articles, those which are mainly or entirely new being distinguished by italics. In a few instances the changes are

merely verbal. The numbers refer to the present edition, but they agree, in most cases, with the numbers in previous editions.

Introduction. Arts. 5, 6, 8, *Def. before 11.*

Forces on a particle. Arts. 28, 32, 34, 35. Props. VIII, X.

Parallel forces. Arts. 40, 41, 50.

Graphical Statics. Chap. III. A.

Forces on a rigid body. Arts. 53, 54, 62, 69, 72, note to 73, *76*, 78, 79, 81, *82*, 88.

Theorems on systems of forces. *Arts. 92—102*, 104, *105*, *Examples* 2, *3*, *4*.

Centre of gravity. Arts. 108, 109, 128, *140—140 F*, 141, *142*, *144 A*.

Mechanical powers. Arts. 145, 149, 150, 152, 156, 168, 170, 181.

Strings. Arts. 190, *192* (*latter part*) *and note*, *193*, *Ex. 38*. Art. 198.

Attraction. Arts. *216* (*first part*), 219, *228—240*, *245—247*, 248 IV. Examples *13*, *14*, *19*, 20, *21*, *22*, *23*.

Work and virtual velocities. *249—271*, 272 *I*, II, III, V, VI, 274.

QUEEN'S COLLEGE, BELFAST.
July, 1887.

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STATICS.

CHAPTER I.

INTRODUCTION.

1. A BODY is a portion of matter limited in every direction, and is consequently of a determinate form and volume. A material particle is a body indefinitely small in every direction; we shall speak of it for shortness as a particle.

2. A body is in motion when the body or its parts occupy successively different positions in space. But we cannot judge of the state of rest or motion of a body without comparing it with other bodies, and for this reason all motions which come under our observation are necessarily *relative* motions.

3. Force is that which produces or tends to produce motion in a body.

4. When several forces act simultaneously on a body, it may happen that they neutralise each other; when a body remains at rest though acted on by forces, it is said to be in equilibrium; or, in other words, the forces are said to maintain equilibrium.

5. Mechanics is the science which treats of the laws of rest and motion of bodies. Statics treats of the laws of the

equilibrium of bodies, and Kinetics of the laws of motion of bodies.

The name *Dynamics* in modern English nomenclature includes both Statics and Kinetics. It was formerly used for Kinetics alone.

6. There are three things to consider in a force acting on a particle: the *position* of the particle: the *direction* of the force, that is, the direction in which it tends to make the particle start; and the *magnitude* of the force. As the dimensions of a particle are indefinitely small, its position may be specified in the same manner as that of a point in geometry, and the direction of the force may be specified in the same manner as that of a straight line in geometry. We proceed then to consider the magnitude or amount of a force.

7. Forces can be measured by taking some force as the unit, and expressing by numbers the ratios which other forces bear to this unit. Two forces are *equal* when being applied in opposite directions to a particle they maintain equilibrium. If we take two *equal* forces and apply them to a particle in the *same* direction we obtain a force *double* of either; if we unite *three* equal forces we obtain a *triple* force; and so on.

When we say then that a force applied to a particle is a certain multiple of another force, we mean that the first force may be supposed to be composed of a certain number of forces equal to the second and all acting in the same direction. In this way forces become measurable quantities, which can be expressed by numbers, like all other quantities, by referring them to a unit of their own kind. Forces may also be represented by straight lines proportional in length to these numbers, drawn from the point at which the forces act and in the directions in which they act.

8. Experience teaches us that if a body be let free from the hand, it will fall downwards in a certain direction; however frequently the experiment be made, the result is the same, the body strikes the same spot on the ground in each

trial, provided the place from which it is dropped remain the same. The cause of this undeviating effect is assumed to be the attraction of the earth, and is termed terrestrial gravitation. If the body be prevented from falling by the interposition of a table or of the hand, the body exerts a *pressure* on the table or hand. The amount of this pressure is called the *weight* of the body.

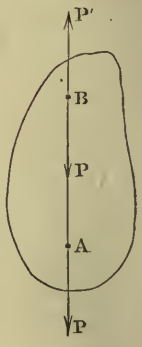
9. A solid body is conceived to be an aggregation of material particles which are held together by their mutual attractions. This appears to be a safe hypothesis, since experiments shew that any body is divisible into successively smaller and smaller portions without limit, if sufficient force be exerted to overcome the mutual action of the parts of the body.

10. A rigid body is one in which the particles retain invariable positions with respect to each other. No body in nature is perfectly rigid; every body yields more or less to the forces which act on it. If, then, in any case this compressibility is of a sensible magnitude, we shall suppose that the body has assumed its figure of equilibrium, and then consider the points of application of the forces as a system of invariable form. By body, hereafter, we mean *rigid* body.

Definition. Whenever in Statics two forces are said to *have the same effect* or to be *equivalent*, the meaning is that any force which would balance one of them would balance the other. A similar meaning is to be understood when one force is said to be equivalent to two or more other forces taken together. When one set of forces is said to be equivalent to another set, the meaning is that any forces which would balance one of the sets would also balance the other.

11. A force acting on a body has the same effect at whatever point of its line of action we suppose it applied, provided this point be either one of the points of the body or be invariably connected with the body. This principle is known by the name of the *transmissibility of a force to any point in its line of action;* it is assumed as an axiom or as an experimental fact. We may shew the amount of assumption involved in the axiom, by the following process.

Suppose a body to be kept in equilibrium by a system of forces, one of which is the force P applied at the point A. Take any point B which lies on the direction of this force, and suppose B so connected with A that the distance AB is unchangeable. Then, if at B we introduce two forces, P and P', equal in magnitude and acting in opposite directions along the straight line AB, it seems evident that no change is made in the effect of the force P at A. Let us now *assume that P at A and P' at B will neutralise each other, and may therefore be removed without disturbing the equilibrium of the body;* then there remains the force P at B producing the same effect as when it acted at A.



12. We shall have occasion hereafter to assume what may be called the *converse* of the principle of the transmissibility of force, namely, that if a force can be transferred from its point of application to a second point without altering its effect, then the second point must be in the direction of the force. See Art. 17.

13. When we find it useful to change the point of application of a force, we shall for shortness not always state that the new point is *invariably connected* with the old point, but this must be always understood.

CHAPTER II.

THE COMPOSITION AND EQUILIBRIUM OF FORCES ACTING ON A PARTICLE.

14. When a particle is acted on by forces which do not maintain equilibrium it will begin to move in some determinate direction. It is clear then that a *single* force may be found of such a magnitude, that if it acted in the direction *opposite* to that in which the motion would take place this force would prevent the motion, and consequently would be in equilibrium with the other forces which act on the particle. If then we were to remove the original forces and replace them by a single force, equal in magnitude to that described above, but acting in the opposite direction, the particle would still remain at rest. This force, which is equivalent in its effect to the combined effect of the original forces, is called their *resultant*, and the original forces are called the *components* of the resultant.

It will be necessary then to begin by deducing rules for the *composition of forces;* that is, for finding their resultant force. After we have determined these, it will be easy to deduce the analytical relations which forces must satisfy when in equilibrium.

15. *To find the resultant of a given number of forces acting on a particle in the same straight line; and to find the condition which they must satisfy that they may be in equilibrium.*

When two or more forces act on a particle in the same direction it is evident that the resultant force is equal to their sum and acts in the same direction.

When two forces act in different directions, but in the same straight line, on a particle, it is equally clear that their resultant is equal to their difference and acts in the direction of the greater component.

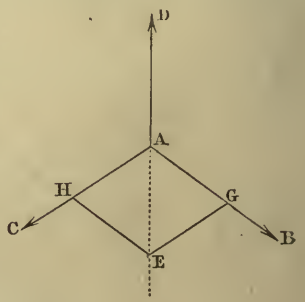
When several forces act in different directions, but in the same straight line, on a particle, the resultant of the forces acting in one direction is equal to the sum of these forces, and acts in the same direction; and so of the forces acting in the opposite direction. The resultant, therefore, of all the forces is equal to the difference of these sums, and acts in the direction of the greater sum.

If the forces acting in one direction are reckoned positive, and those in the opposite direction negative, then their resultant is equal to their *algebraical* sum; its sign determines the direction in which it acts.

In order that the forces may be in equilibrium, their resultant, and therefore their algebraical sum, must vanish.

16. There is another case in which we can easily determine the magnitude and direction of the resultant.

Let AB, AC, AD be the directions of three equal forces acting on the particle A; suppose these forces all in the same plane and the three angles BAC, CAD, DAB each equal to 120°; the particle will remain at rest, for there is no reason why it should move in one direction rather than another. Each of the forces is therefore equal and opposite to the resultant of the other two. But if we take on the directions of two of them, AB, AC, two equal straight lines AG, AH to represent the forces, and complete the parallelogram $GAHE$, the diagonal AE will lie in the same straight line with AD. Also the triangle AGE will be equilateral, and therefore $AE = AG$. Hence, the diagonal AE of the parallelogram constructed on AG, AH represents the resultant of the two forces which AG and AH respectively represent.



This proposition is a particular case of one to which we now proceed.

17. *If two forces acting at a point be represented in direction and magnitude by two straight lines drawn from the point, and a parallelogram be described on these straight lines as adjacent sides, then the resultant will be represented in direction and magnitude by that diagonal of the parallelogram which passes through the point.*

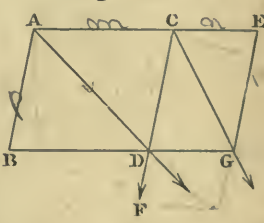
This Proposition is called the *Parallelogram of Forces.*

I. To find the *direction* of the resultant.

When the forces are equal it is clear that the direction of the resultant will *bisect* the angle between the directions of the forces; or, if we represent the forces in magnitude and direction by two straight lines drawn from the point where they act, and describe a parallelogram on these straight lines, that diagonal of the parallelogram which passes through the point will be the direction of the resultant.

Let us suppose that this is true for forces p and m inclined at any angle, and also for forces p and n inclined at the same angle; we can shew that it must then be true for two forces p and $m+n$ also inclined at the same angle.

Let A be the point at which the forces p and m act; AB, AC their directions and proportional to them in magnitude: complete the parallelogram BC, and draw the diagonal AD; then, by hypothesis, the resultant of p and m acts along AD.



Again, take CE in the same ratio to AC that n bears to m. By Art. 11 we may suppose the force n which acts in the direction AE to be applied at A or C; and therefore the forces p, m, and n, in the straight lines AB, AC, and CE, are the same as p and $m+n$ in the straight lines AB and AE.

Now replace p and m by their resultant and transfer its point of application from A to D; then resolve this force at D into two parallel to AB and AC respectively; these resolved parts must evidently be p and m, the former acting in the direction DF, and the latter in the direction DG. Then transfer p to C and m to G.

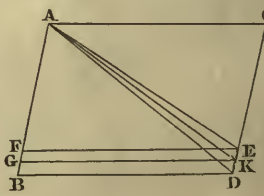
By the hypothesis, p and n acting at C have a resultant in the direction CG; therefore p and n may be replaced by their resultant and its point of application transferred to G. And m has also been transferred to G. Hence by this process we have removed the forces which acted at A to the point G without altering their effect. We may infer then (see Art. 12) that G is a point in the direction of the resultant of p and $m+n$ at A; that is, the resultant of p and $m+n$ acts in the direction of the diagonal AG, provided the hypothesis is correct. But the hypothesis is correct for equal forces, as p, p, and therefore it is true for forces p, $2p$; consequently for p, $3p$, and so on; hence it is true for p, $r.p$.

Hence it is true for p, $r.p$, and p, $r.p$, and consequently for $2p$, $r.p$, and so on; and it is finally true for $s.p$ and $r.p$, where r and s are positive integers.

We have still to shew that the Proposition is true for *incommensurable* forces.

This may be inferred from the fact that when two magnitudes are incommensurable, so that the ratio of one to the other cannot be expressed *exactly* by a fraction, we can still find a fraction which differs from the true ratio by a fraction less than any assigned fraction. Or it may be established indirectly thus.

Let AB, AC represent two such forces. Complete the parallelogram BC. Then if their resultant do not act along AD suppose it to act along AE; draw EF parallel to AC. Divide AC into a number of equal portions, each less than DE; mark off from CD portions equal to these, and let K be the last division; this evidently falls between D and E; draw GK parallel to AC. Then two forces represented by AC, AG have a resultant in the direction AK, because they are commensurable; therefore the forces AC and AB are equivalent to a force along AK together with a force equal to GB applied at A along AB. And we may assume as obvious that the resultant of these forces must lie

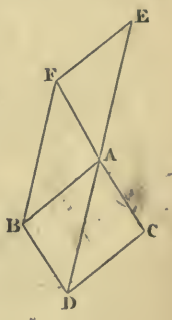


between AK and AB; but by supposition the resultant is AE which is *not* between AK and AB. This is absurd.

In the same manner we may shew that every direction except AD leads to an absurdity, and therefore the resultant must act along AD, whether the forces be commensurable or incommensurable.

II. To find the *magnitude* of the resultant.

Let AB, AC be the directions of the given forces, AD that of their resultant; take AE opposite to AD, and of such a length as to represent the *magnitude* of the resultant. Then the forces represented by AB, AC, AE, balance each other. On AE and AB as adjacent sides construct the parallelogram $ABFE$; then the diagonal AF is the direction of the resultant of AE and AB.



Hence AC is in the same straight line with AF; hence FD is a parallelogram; and therefore $AE = FB = AD$. Hence the resultant is represented in *magnitude* as well as in direction by the diagonal of the parallelogram.

Thus the proposition called the *Parallelogram of Forces* is completely established.

18. Hence if P and Q represent two component forces acting at an angle α on a particle, the resultant R is given by the equation

$$R^2 = P^2 + Q^2 + 2PQ\cos\alpha.$$

19. *When three forces acting on a particle are in equilibrium each is proportional to the sine of the angle included by the directions of the other two.*

For if we refer to the third figure of Art. 17 we have

$$P : Q : R :: AB : AC \text{ (or } BD) : AD$$
$$:: \sin ADB : \sin BAD : \sin ABD$$
$$:: \sin CAE : \sin BAE : \sin BAC.$$

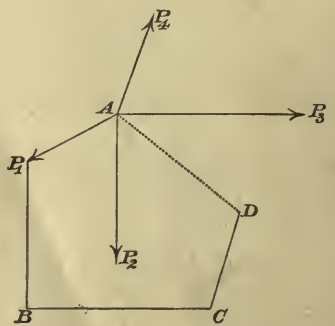
Conversely if three forces act on a particle, and each force is as the sine of the angle between the directions of the other two, it may be shewn that one of the forces is equal in *magnitude* to the resultant of the other two, and acts either in the *same* direction or in the *opposite* direction: in the latter case the three forces are in equilibrium.

It should be noticed that if the sides of a triangle be drawn parallel to the directions of the forces, the length of any side will be proportional to the sine of the angle between the forces which correspond to the other two sides.

20. Any force acting on a particle may be replaced by two others, if the sides of a triangle drawn parallel to the directions of the forces have the same relative proportion that the forces have. For by the parallelogram of forces the resultant of the latter two forces is equal to the given force.

This is called the *resolution* of a force.

21. Since the resultant of two forces acting on a particle is represented in magnitude and direction by the diagonal of the parallelogram constructed upon the straight lines which represent these forces in magnitude and direction, it follows that, in order to obtain the resultant of the forces $P_1, P_2, P_3, \ldots$ which act on a particle A, and are represented by the straight lines $AP_1, AP_2, AP_3, \ldots$ we may proceed thus:



Find the resultant of P_1 and P_2, compound this resultant with P_3, this new resultant with P_4, and so on. It follows

from this, that if we construct a polygon AP_1BCD, of which the sides are respectively equal and parallel to the straight lines AP_1, AP_2,... and join A with the last vertex D, the straight line AD will represent in magnitude and direction the resultant of all the forces.

We may conclude that the necessary and sufficient condition for the equilibrium of a number of forces acting on a particle is, that the point D should coincide with A; that is, that the figure $AP_1B...D$ should be a *complete* polygon. The forces in the figure are not necessarily all in one plane.

The result here obtained may be enunciated thus: *If the sides of any polygon taken in order are respectively proportional to the magnitudes of forces acting at a point, and parallel to the directions of the forces, then the forces will be in equilibrium.*

This proposition is called the *Polygon of Forces.*

The student must carefully notice the conditions under which this proposition is asserted to hold; the forces are supposed all to act *at one point,* and are to be represented by the sides of a polygon *taken in order.* As an example of the latter condition, suppose a quadrilateral $ABCD$; then if forces which may be represented by AB, BC, CD, DA, act at a point the forces will be in equilibrium: but the forces will not be in equilibrium if represented by AB, BC, DC, DA, or by AB, BC, CD, AD.

The direction and magnitude of the resultant may also be determined analytically, as in the following Articles.

22. *Any number of forces act on a particle in one plane; required to find the magnitude and the direction of their resultant.*

Let P_1, P_2, P_3,... be the forces, and α_1, α_2, α_3,... the angles their directions make with a fixed straight line drawn through the proposed point. Take this fixed straight line for the axis of x, and one perpendicular to it for that of y. Then, by Art. 20, P_1 may be resolved into $P_1 \cos \alpha_1$, and $P_1 \sin \alpha_1$, acting along the axes of x and y respectively. The other forces may

be similarly resolved. By algebraical addition of the forces which act in the same straight line we have

$$P_1 \cos \alpha_1 + P_2 \cos \alpha_2 + P_3 \cos \alpha_3 + \ldots \text{ along the axis of } x,$$
$$P_1 \sin \alpha_1 + P_2 \sin \alpha_2 + P_3 \sin \alpha_3 + \ldots \text{ along the axis of } y.$$

We shall express the former by $\Sigma P \cos \alpha$ and the latter by $\Sigma P \sin \alpha$, where the symbol Σ denotes that we take the sum of all the quantities of which the quantity before which it is placed is the type.

If we put $P_1 \cos \alpha_1 = X_1$ and $P_1 \sin \alpha_1 = Y_1$, and use a similar notation for the other components, we have two forces replacing the whole system, namely ΣX along the axis of x and ΣY along that of Y. If R denote the resultant of these forces and a the angle at which it is inclined to the axis of x, we have, by Art. 17,

$$R^2 = (\Sigma X)^2 + (\Sigma Y)^2, \quad \tan a = \frac{\Sigma Y}{\Sigma X}.$$

Also
$$\cos a = \frac{\Sigma X}{R}; \quad \sin a = \frac{\Sigma Y}{R}.$$

23. *To find the conditions of equilibrium when any number of forces act on a particle in one plane.*

When the forces are in equilibrium we must have $R = 0$; therefore

$$(\Sigma X)^2 + (\Sigma Y)^2 = 0;$$

therefore
$$\Sigma X = 0; \quad \Sigma Y = 0;$$

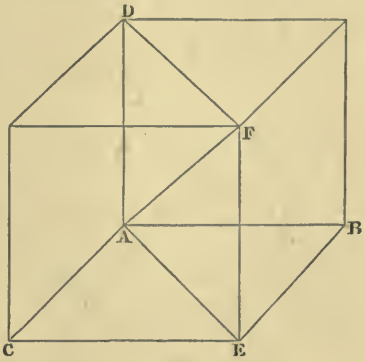
and these are the conditions among the forces that they may be in equilibrium.

Hence *a system of forces acting in one plane on a particle will be in equilibrium if the sums of the resolved parts of the forces along two rectangular axes in the plane through the point vanish.*

Conversely, if the forces are in equilibrium the sum of the resolved parts of the forces along any straight line through the point will vanish.

24. *Three forces act on a particle in directions making right angles with each other; required to find the magnitude and direction of their resultant.*

Let AB, AC, AD represent the three forces X, Y, Z in magnitude and direction. Complete the parallelogram BC, and draw the diagonal AE; then AE represents the resultant of X and Y in magnitude and direction, by Art. 17. Now



the resultant of this force and Z, that is, of the forces represented by AE, AD, is represented in magnitude and direction by AF, the diagonal of the parallelogram DE. Hence the resultant of X, Y, Z is represented in magnitude and direction by AF. Let R be the magnitude of the resultant, and a, b, c the angles the direction of R makes with those of X, Y, Z. Then, since

$$AF^2 = AE^2 + AD^2 = AB^2 + AC^2 + AD^2,$$

therefore

$$R^2 = X^2 + Y^2 + Z^2.$$

Also $\cos a = \dfrac{AB}{AF} = \dfrac{X}{R}$, $\cos b = \dfrac{AC}{AF} = \dfrac{Y}{R}$, $\cos c = \dfrac{AD}{AF} = \dfrac{Z}{R}$.

Thus the magnitude and direction of the resultant are determined.

25. It follows from the last Article that any force R the direction of which makes the angles a, b, c with three rectangular axes fixed in space, may be replaced by the three forces $R\cos a$, $R\cos b$, $R\cos c$, acting simultaneously on the particle on which R acts, and having their directions parallel to the axes of coordinates respectively.

26. *Any number of forces act on a particle in any directions; required to find the magnitude and the direction of their resultant.*

Let P_1, P_2, P_3,... be the forces; let α_1, β_1, γ_1 be the angles which the direction of P_1 makes with three rectangular axes drawn through the proposed point; let α_2, β_2, γ_2 be the angles which the direction of P_2 makes with the same axes; and so on.

Then, by Art. 25, the components of P_1 in the directions of the axes are

$$P_1 \cos \alpha_1,\ P_1 \cos \beta_1,\ P_1 \cos \gamma_1,\ \text{(or } X_1,\ Y_1,\ Z_1,\ \text{suppose)}.$$

Resolve each of the other forces in the same way, and reduce the system to three forces, by adding those which act in the same straight line, Art. 15; we thus have

$$P_1 \cos \alpha_1 + P_2 \cos \alpha_2 + \ldots \text{ or } \Sigma P \cos \alpha, \text{ or } \Sigma X,$$

$$P_1 \cos \beta_1 + P_2 \cos \beta_2 + \ldots \text{ or } \Sigma P \cos \beta, \text{ or } \Sigma Y,$$

$$P_1 \cos \gamma_1 + P_2 \cos \gamma_2 + \ldots \text{ or } \Sigma P \cos \gamma, \text{ or } \Sigma Z,$$

acting in the directions of the axes of x, y, and z respectively.

If we call the resultant R, and the angles which its direction makes with the axes a, b, c, we have, by Art. 24,

$$R^2 = (\Sigma X)^2 + (\Sigma Y)^2 + (\Sigma Z)^2,$$

and $$\cos a = \frac{\Sigma X}{R},\ \cos b = \frac{\Sigma Y}{R},\ \cos c = \frac{\Sigma Z}{R}.$$

27. *To find the conditions of equilibrium when any number of forces act on a particle.*

When the forces are in equilibrium, we must have $R = 0$; therefore

$$(\Sigma X)^2 + (\Sigma Y)^2 + (\Sigma Z)^2 = 0,$$

therefore $$\Sigma X = 0;\ \Sigma Y = 0;\ \Sigma Z = 0;$$

and these are the conditions among the forces that they may be in equilibrium.

28. The expression for the magnitude of the resultant in Art. 26 may be rendered independent of the position of the axes. For, from Art. 26,

$$R^2 = (P_1 \cos\alpha_1 + P_2 \cos\alpha_2 + \ldots)^2 + (P_1 \cos\beta_1 + P_2 \cos\beta_2 + \ldots)^2 + (P_1 \cos\gamma_1 + P_2 \cos\gamma_2 + \ldots)^2.$$

When the expressions on the right-hand side are developed, we shall find that the coefficient of P_1^2 is

$$\cos^2\alpha_1 + \cos^2\beta_1 + \cos^2\gamma_1,$$

and that the coefficient of $P_1 P_2$ is

$$2(\cos\alpha_1 \cos\alpha_2 + \cos\beta_1 \cos\beta_2 + \cos\gamma_1 \cos\gamma_2).$$

Now we know from Analytical Geometry of three dimensions that

$$\cos^2\alpha_1 + \cos^2\beta_1 + \cos^2\gamma_1 = 1;$$

and that

$$\cos\alpha_1 \cos\alpha_2 + \cos\beta_1 \cos\beta_2 + \cos\gamma_1 \cos\gamma_2$$

is equal to the cosine of the angle between the directions of the forces P_1 and P_2, which we may denote by $\cos(P_1, P_2)$. Similar values will be found for the coefficients of the other terms; and the result may be expressed thus,

$$R^2 = \Sigma P^2 + 2\Sigma PP' \cos(P, P'),$$

where by P, P' we mean any two of the forces.

Again let S_1 denote the total component in the direction of P_1, with similar notation for the other forces; the value of R^2 just obtained can be written

$$\begin{aligned} R^2 &= P_1\{P_1 + P_2\cos(P_1P_2) + \ldots\} + P_2\{P_2 + P_1\cos(P_2P_1) + \ldots\} \\ &\quad + P_3\{P_3 + P_1\cos(P_3P_1) + \ldots\} + \ldots\ldots \\ &= P_1S_1 + P_2S_2 + P_3S_3 + \ldots\ldots = \Sigma PS. \end{aligned}$$

29. The equation $R\cos\alpha = \Sigma P\cos\alpha$, in Art. 26, shews that the *resolved part of the resultant in any direction is equal to the sum of the resolved parts of the components in the same direction;* for since the axes were taken arbitrarily, that of x might have been made to coincide with any assigned direction. Or we may establish the proposition thus. Suppose

a straight line drawn through the point of application of the forces, and inclined to the axes at angles α', β', γ'. Take the three equations of Art. 26,

$$R\cos a = P_1\cos\alpha_1 + P_2\cos\alpha_2 + \ldots\ldots$$
$$R\cos b = P_1\cos\beta_1 + P_2\cos\beta_2 + \ldots\ldots$$
$$R\cos c = P_1\cos\gamma_1 + P_2\cos\gamma_2 + \ldots\ldots$$

Multiply the first by $\cos\alpha'$, the second by $\cos\beta'$, and the third by $\cos\gamma'$, and add. Then, if θ_1, θ_2... denote the angles which P_1, P_2... make with the arbitrarily drawn straight line, and θ the angle which the resultant R makes with it, we have, by the formula quoted in Art. 28 for the cosine of the angle between two straight lines,

$$R\cos\theta = P_1\cos\theta_1 + P_2\cos\theta_2 + \ldots\ldots$$

30. From Art. 20 it is obvious that a given force may be resolved into two others in an infinite number of ways. When we speak of the resolved part of a force in a given direction, as in the preceding Article, *we shall always suppose, unless the contrary is expressed,* that the given force is resolved into two forces, one in the *given direction* and the other in a direction *at right angles to the given direction.* The former component we shall call the resolved force in the given direction.

When forces act on a particle it will be in equilibrium, provided the sums of the forces resolved along three directions not lying in one plane are zero. For if the forces do not balance, they must have a single resultant; and as a straight line cannot be at right angles to three straight lines which meet at a point and are not in the same plane, the resolved part of the resultant, and therefore the sum of the resolved parts of the given forces, along these three straight lines, could not vanish, which is contrary to the hypothesis.

31. In Art. 26 we resolved each force of a system into three others along three *rectangular* axes. In the same way we may, if we please, resolve each force along three straight lines forming a *system of oblique axes.* For whether the figure in Art. 24 represent an *oblique* or *rectangular* parallelepiped, the force AF may be resolved into AD and AE, and

the latter again resolved into AB and AC. Hence the resultant of a system of forces may be represented by the diagonal of an oblique parallelepiped, and for equilibrium it will be necessary that this diagonal should vanish, and therefore that the edges of the parallelepiped should vanish.

The following three Articles are particular cases of the equilibrium of a particle.

32. *To determine the condition of equilibrium of a particle acted on by any forces and constrained to remain on a given smooth curve.*

By a *smooth* curve we understand a curve that can only exert force on the particle in a direction *normal* to the curve at the point of contact.

Let X, Y, Z denote the forces acting on the particle in directions parallel to three rectangular axes, exclusive of the action of the curve. Let x, y, z denote the co-ordinates of the particle, and s the length of the arc measured from some fixed point up to the point (x, y, z). Then by Analytical Geometry of three dimensions the cosines of the angles which the tangent to the curve at the point (x, y, z) makes with the axes are $\frac{dx}{ds}$, $\frac{dy}{ds}$, $\frac{dz}{ds}$, respectively. The forces acting on the particle being resolved along the tangent to the curve, their sum is

$$X\frac{dx}{ds} + Y\frac{dy}{ds} + Z\frac{dz}{ds}.$$

Unless this vanishes, there will be nothing to prevent the particle from moving; for equilibrium then we must have

$$X\frac{dx}{ds} + Y\frac{dy}{ds} + Z\frac{dz}{ds} = 0.$$

Conversely if this relation holds the particle will remain at rest, for there is no force to make it move *along the curve*, which is the only motion of which it is capable.

Another statement of the condition of equilibrium is

$$R\cos(R, ds) = 0,$$

where R denotes the resultant of the given forces, and (R, ds) the inclination of this resultant to the tangent. The two statements can easily be shewn to be equivalent, since the value of cos (R, ds) is

$$\frac{X}{R}\frac{dx}{ds} + \frac{Y}{R}\frac{dy}{ds} + \frac{Z}{R}\frac{dz}{ds}.$$

We have supposed the particle to be placed inside a tube which has the form of the curve. If, however, the particle be merely placed in *contact with a curve*, it will be *further* necessary for equilibrium that the resultant of the forces should press the particle *against* the curve and not move it *from* the curve.

33. *To determine the conditions of equilibrium of a particle acted on by any forces and constrained to remain on a given smooth surface.*

A *smooth* surface is one which can exert no force on the particle except in a direction normal to the surface.

Let X, Y, Z denote the forces acting on the particle in directions parallel to three rectangular axes, exclusive of the action of the surface, and let R denote their resultant. R must act in a direction normal to the surface at the point where the particle is situated; for if it did not, we might decompose it into two forces, one in the normal and one at right angles to the normal, of which the latter would set the particle in motion. The cosines of the angles which R makes with the axes are proportional to X, Y, Z respectively; and if $F(x, y, z) = 0$ be the equation to the surface, the cosines of the angles which the normal to the surface at the point (x, y, z) makes with the axes, are by Analytical Geometry of three dimensions proportional to $\frac{dF}{dx}$, $\frac{dF}{dy}$, and $\frac{dF}{dz}$ respectively. Hence for equilibrium we must have

$$\frac{X}{\frac{dF}{dx}} = \frac{Y}{\frac{dF}{dy}} = \frac{Z}{\frac{dF}{dz}}.$$

If these relations are satisfied, the resultant force is directed along the normal; hence, if we suppose the particle incapable of leaving the surface, the above conditions will be *sufficient* to ensure its equilibrium; but if the particle be merely placed *on* a surface, it will be further necessary that R should be directed *towards* not *from* the surface. For example, if the particle be placed on the outside of a sphere, R must be directed *towards* the centre of the sphere.

In both the preceding cases the reaction of the curve or surface is equal and opposite to the resultant R of the given forces.

34. Duchayla's proof of the Parallelogram of Forces, which we have given in Art. 17, rests on the principle of the *transmissibility of force;* see Art. 11. The following proof does not involve this principle. The first part of it is taken from a proof of the Parallelogram of Forces given by Laplace (*Mécanique Celeste,* Liv. I. Chap. 1).

Let x and y denote two forces which are inclined at a right angle, and let z denote their resultant; we propose to find the value of z. It is obvious that if the components instead of being x and y were $2x$ and $2y$ respectively, the resultant would be $2z$ and would have the same direction as before; so if the components were $3x$ and $3y$ respectively, the resultant would be $3z$ and would have the same direction as before; and so on. We may therefore assume conversely, that if the inclination of the resultant to each component remains unchanged, the ratio of each component to the resultant will also remain unchanged. Now consider the force x as the resultant of two forces x' and x'', of which x' is in the direction of z, and x'' is at right angles to that direction. Then by the principle just assumed, we have

$$\frac{x'}{x} = \frac{x}{z}, \text{ and } \frac{x''}{x} = \frac{y}{z};$$

so that

$$x' = \frac{x^2}{z}, \text{ and } x'' = \frac{xy}{z}.$$

Similarly y may be resolved into $\frac{y^2}{z}$ along the direction of z

and $\frac{yx}{z}$ at right angles to that direction. Thus the forces x and y are equivalent to four forces, two in the direction of z and the other two at right angles to that direction; the latter two are equal in magnitude and opposite in direction, so that they counteract each other; hence the resultant of the former two must be equal to z. Thus

$$\frac{x^2}{z} + \frac{y^2}{z} = z \text{; therefore } z^2 = x^2 + y^2.$$

This proves that when two forces are at right angles, the Parallelogram of Forces is true as regards the magnitude of their resultant; we have next to shew that it is true as regards direction.

35. This proposition is obvious in the case where the components P and Q are equal; for then the resultant must necessarily bisect the angle between the directions of P and Q, and will therefore coincide with the diagonal.

It may next be shewn to be true in the case where $Q = P\sqrt{2}$. For consider three forces, each equal to P, acting in directions mutually at right angles. The forces may be represented by the three edges of a cube which meet at a point. The resultant of two of the forces, by what we have already shewn, is $P\sqrt{2}$, and acts along the diagonal of that face of the cube of which these two forces are edges. Hence the resultant of the three forces must lie in the plane which passes through this diagonal and that edge of the cube which is at right angles to this face. In a similar manner another plane can be assigned in which the resultant of the three forces must lie, by beginning with another pair out of the three equal forces. Therefore the resultant of the three equal forces must lie in the common intersection of the two planes, that is in the diagonal of the cube which passes through the point of application of the forces. But this resultant is also the resultant of two forces P and $P\sqrt{2}$, which act at right angles; and the diagonal of the cube coincides with the diagonal of the rectangle constructed on the sides which represent these forces. Thus the proposition is true in the case where $Q = P\sqrt{2}$.

We shall now shew that if the proposition is true when $Q = P\sqrt{m}$ it is true when $Q = P\sqrt{(m+1)}$. For consider three forces P, P, and $P\sqrt{m}$, acting in directions mutually at right angles. The forces may be represented by the three edges of a rectangular parallelepiped which meet at a point. Then by first compounding $P\sqrt{m}$ with one of the forces P, we see that the resultant of the three forces must lie in the plane passing through the diagonal of the rectangle corresponding to the first two forces, and also through the edge which corresponds to the other force P. In a similar manner another plane can be assigned in which the resultant of the three forces must lie. Therefore the resultant of the three forces must lie in the common intersection of the two planes, that is in the diagonal of the rectangular parallelepiped which passes through the point of application of the forces. But this resultant is also the resultant of the two forces P and $P\sqrt{(m+1)}$ which act at right angles; and the diagonal of the rectangular parallelepiped coincides with the diagonal of the rectangle constructed on the sides which represent these forces. Thus the proposition is true in the case where $Q = P\sqrt{(m+1)}$.

Hence we see that the proposition is true when $Q = P\sqrt{m}$, where m is any integer.

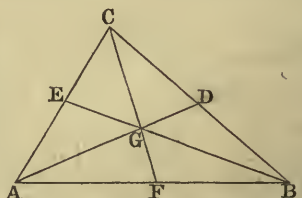
Finally consider three forces P, $P\sqrt{m}$, and $P\sqrt{n}$, where m and n are any integers, acting in directions mutually at right angles. The forces may be represented by the three edges of a rectangular parallelepiped which meet at a point. Then the resultant of the three forces must lie in the plane which contains the edge $P\sqrt{n}$, and the diagonal $P\sqrt{(m+1)}$; it must also lie in the plane which contains the edge $P\sqrt{m}$, and the diagonal $P\sqrt{(n+1)}$. Hence the resultant of the three forces must lie in the common intersection of the two planes, that is, in the diagonal of the rectangular parallelepiped which passes through the point of application of the forces. Hence the resultant of $P\sqrt{m}$ and $P\sqrt{n}$ must lie in the plane which contains the edge P, and the diagonal of the rectangular parallelepiped; and this plane cuts the plane of $P\sqrt{m}$ and $P\sqrt{n}$ in the diagonal of the face which corresponds to these edges. Therefore the direction of the resultant of the forces $P\sqrt{m}$ and $P\sqrt{n}$ is the diagonal of the rectangle constructed

on the sides which represent these forces. Hence the proposition is true for forces which are in the ratio of $P\sqrt{m}$ to $P\sqrt{n}$, that is, in the ratio of P to $P\sqrt{\frac{n}{m}}$, where m and n are any integers. But by properly choosing m and n we may make this as close as we please to the ratio between any two assigned forces. Therefore the proposition is true whatever may be the ratio between the forces.

We can now prove the Parallelogram of Forces for two forces at any angle. Let OA, OB represent the two forces, and OC the diagonal of the parallelogram of which OA, OB are sides. Resolving the two forces along and perpendicular to OC, the components perpendicular to OC, being represented by the perpendicular distances of A and B from OC, are equal and destroy one another; and the components along OC, being represented by the projections of OA and AC upon OC, are together represented by OC.

36. We shall now give some simple propositions which will serve to exemplify and illustrate the principles of the present Chapter.

I. ABC is a triangle; D, E, F are the middle points of the sides BC, CA, AB respectively: shew that forces represented by the straight lines AD, BE, CF will be in equilibrium.



It is known that the straight lines AD, BE, CF meet at a point: see *Appendix to Euclid*. Let G denote this point. The three forces may be supposed to act at G.

Since D is the middle point of BC, the parallelogram described on AB and AC as adjacent sides will have a diagonal in the direction AD; hence twice AD will represent the resultant of two forces represented by AB and AC. And conversely the force represented by AD may be resolved into two forces represented by half AB and half AC. Similarly the force BE may be resolved into half BC and half BA; and the force CF may be resolved into half CA and half CB.

But the force half AB is equal and opposite to the force half BA; and so on. Thus, finally, the forces AD, BE, CF are in equilibrium.

II. In the figure of the preceding proposition forces represented by the straight lines GA, GB, GC will be in equilibrium.

The resultant of the forces GB and GC acts along GD. If then there is not equilibrium the three forces GA, GB, GC have a resultant acting either from A towards D or from D towards A, that is in the straight line AD. But in the same way it may be shewn that if the forces GA, GB, GC are not in equilibrium their resultant must act in the straight line BE, and also in the straight line CF. But it is impossible that the resultant can act in three different straight lines. Therefore the forces GA, GB, GC must be in equilibrium.

As the student is probably aware, it may be shewn by Geometry that AG is equal to twice GD; and thus the present theorem may be established directly; but we have used the method here given for the purpose of illustrating mechanical principles. We may observe that we have thus by the aid of mechanical principles, in fact, demonstrated that $AG = 2GD$; for the resultant of GB and GC is represented by twice GD.

Since $AD = 3GD$, $BE = 3GE$, and $CF = 3GF$, the forces AD, BE, CF have the *same relative proportion* as the forces GD, GE, GF; so that the first proposition may be deduced immediately from the second.

III. Forces act at the middle points of the sides of a triangle, in the plane of the triangle, at right angles to the sides and respectively proportional to the sides: shew that if they all act inwards or all act outwards they will be in equilibrium.

The directions of the forces meet at a point, namely, the centre of the circle which circumscribes the triangle. And the angle between the directions of two forces is the supplement of the angle between the corresponding sides. Thus each force is as the sine of the angle between the other two. Hence by Art. 19 the forces are in equilibrium.

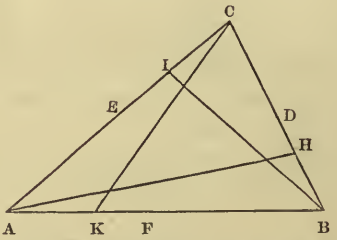
IV. Forces act at the angular points of a triangle along the perpendiculars drawn from the angular points on the respectively opposite sides; and the forces are respectively proportional to the sides: shew that the forces will be in equilibrium.

It is known that the perpendiculars meet at a point: see *Appendix to Euclid.* Hence by the same method as in the preceding proposition the forces are in equilibrium.

V. ABC is a triangle; H, I, K are points in the sides BC, CA, AB respectively such that

$$\frac{BH}{HC} = \frac{CI}{IA} = \frac{AK}{KB}.$$

Shew that if forces represented by AH, BI, CK act at a point they will be in equilibrium.



Let D, E, F be the middle points of the sides; and suppose $AD, BE,$ and CF to be drawn.

The force AH may be resolved into the forces AD, DH; the force BI into the forces BE, EI; and the force CK into the forces CF, FK. See Art. 20.

The forces AD, BE, CF are in equilibrium by the first proposition.

And we have from the hypothesis as to H, I, K,

$$\frac{DH}{BC} = \frac{EI}{CA} = \frac{FK}{AB};$$

so that the forces DH, EI, FK are proportional to the sides of the triangle ABC taken in order; and they are therefore in equilibrium by Art. 21 if they act at a point.

Hence if the forces AH, BI, CK act at a point they are in equilibrium.

The straight lines AH, BI, CK by their intersections form a triangle; and therefore by Art. 19 the sides of this triangle are proportional to the forces. Hence we arrive by mechanical principles at the following geometrical result: the sides of the triangle formed by the intersections of AH, BI, CK are proportional to AH, BI, CK respectively.

VI. A, B, C are three points on the circumference of a circle; forces act along AB and BC inversely proportional to these straight lines in magnitude: shew that the resultant acts along the tangent at B.

Denote the forces by $\frac{\mu}{AB}$ and $\frac{\mu}{BC}$ respectively. Resolve them at right angles to the tangent at B; thus we obtain by Euclid, III. 32,

$$\frac{\mu}{AB}\sin ACB - \frac{\mu}{BC}\sin CAB;$$

and this is zero, since

$$\frac{BC}{AB} = \frac{\sin CAB}{\sin ACB}.$$

Hence the resultant must act along the tangent at B.

VII. If one of two forces be given in magnitude and position, and also the direction of their resultant, the locus of the extremity of the straight line representing the other force will be a straight line.

Let a and r denote the magnitudes of the two forces; suppose the former to make an angle α with the direction of the resultant, and the latter an angle θ.

Then, resolving along the straight line which is at right angles to the direction of the resultant, we have

$$a\sin\alpha - r\sin\theta = 0.$$

Now a and α being given, while r and θ are variable, this equation represents a straight line which is parallel to the

direction of the resultant, and at a distance $a \sin \alpha$ from it. See *Conic Sections*, Chap. II.

VIII. From any point within a regular polygon perpendiculars are drawn on all the sides of the polygon: shew that the direction of the resultant of all the forces represented by these perpendiculars passes through the centre of the circle circumscribing the polygon, and find the magnitude of the resultant.

Let O be the centre, C the given point, $\alpha_1, \alpha_2, \ldots \alpha_n$ the inclinations of OC to the perpendiculars from O on the sides. Let p denote the length of one of these perpendiculars, and c the distance OC.

The magnitudes of the given forces are $p - c\cos\alpha_1$, $p - c\cos\alpha_2, \ldots p - c\cos\alpha_n$, and the n forces p can be neglected, since from the symmetry of their arrangement they form a system in equilibrium. The resultant of the given forces will therefore be the resultant of forces $c\cos\alpha_1$, $c\cos\alpha_2, \ldots c\cos\alpha_n$ in the opposite directions to the perpendiculars from C on the sides. Resolving along and perpendicular to CO, the total force along CO is

$$c\,(\cos^2\alpha_1 + \cos^2\alpha_2 + \ldots + \cos^2\alpha_n)$$

$$= n\,\frac{c}{2} + \frac{c}{2}\,(\cos 2\alpha_1 + \cos 2\alpha_2 + \ldots + \cos 2\alpha_n).$$

But the series in the brackets vanishes (see *Trigonometry*, Chap. XXII.), hence the total force along CO is $n\,\frac{c}{2}$.

The total force perpendicular to CO is

$$c\,(\cos\alpha_1 \sin\alpha_1 + \cos\alpha_2 \sin\alpha_2 + \ldots + \cos\alpha_n \sin\alpha_n)$$

$$= \frac{c}{2}\,(\sin 2\alpha_1 + \sin 2\alpha_2 + \ldots + \sin 2\alpha_n),$$

which vanishes. The resultant is therefore $\frac{n}{2}\,CO$.

IX. Suppose three forces P, Q, R to act in one plane at a point O, and to be in equilibrium; let a circle be de-

scribed with O as centre, and any radius, cutting the directions of the forces at the points A, B, C respectively: then shall P, Q, R be respectively proportional to the areas of the triangles OBC, OCA, OAB.

This follows at once from Art. 19, since the area of a triangle is expressed by half the product of two sides into the sine of the included angle.

Conversely, if P, Q, R are respectively proportional to these areas, they will be in equilibrium.

X. Suppose four forces P, Q, R, S to act at a point O, and to be in equilibrium; let a sphere be described with O as centre, and any radius, cutting the directions of the forces at the points A, B, C, D respectively: then shall P, Q, R, S be respectively proportional to the volumes of the pyramids $OBCD$, $OCDA$, $ODAB$, $OABC$.

Take O as the origin of a set of rectangular axes, let x_1, y_1, z_1 be the co-ordinates of A; x_2, y_2, z_2 the co-ordinates of B; and so on. Then, by Art. 27,

$$\begin{aligned} Px_1 + Qx_2 + Rx_3 + Sx_4 &= 0, \\ Py_1 + Qy_2 + Ry_3 + Sy_4 &= 0, \\ Pz_1 + Qz_2 + Rz_3 + Sz_4 &= 0. \end{aligned}$$

These equations give as the values of P, Q, R, S any equimultiples of the following values

$$P = \begin{vmatrix} x_2 & x_3 & x_4 \\ y_2 & y_3 & y_4 \\ z_2 & z_3 & z_4 \end{vmatrix}, \quad Q = -\begin{vmatrix} x_3 & x_4 & x_1 \\ y_3 & y_4 & y_1 \\ z_3 & z_4 & z_1 \end{vmatrix}, \text{ \&c.}$$

and these determinants are the expressions (as is shewn in works on Coordinate Geometry of three dimensions) for six times the volumes of the pyramids $OBCD$, $OCDA$, &c.

Conversely, if P, Q, R, S are respectively proportional to these volumes they will be in equilibrium.

EXAMPLES.

1. Two forces P and Q have a resultant R which makes an angle α with P; if P be increased by R while Q remains unchanged, shew that the new resultant makes an angle $\frac{\alpha}{2}$ with P.

2. Two forces in the ratio of 2 to $\sqrt{3}-1$, are inclined to each other at an angle of 60^0; what must be the direction and magnitude of a third force which produces equilibrium?

Result. The required force must be to the first of the given forces as $\sqrt{6}$ to 2; and its direction produced makes an angle of 15^0 with that force.

3. The resultant of two forces P and Q is equal to $Q\sqrt{3}$, and makes an angle of 30^0 with P; find P in terms of Q.

Result. Either $P=Q$ or $P=2Q$; in the former case the angle between P and Q is 60^0, in the latter 120^0.

4. If D, E, F be the middle points of the sides of the triangle ABC and O any other point, shew that the system of forces represented by OD, OE, OF is equivalent to that represented by OA, OB, OC.

5. The resultant of two forces is 10 lbs., one of them is equal to 8 lbs., and the direction of the other is inclined to the resultant at an angle of 36^0. Find the angle between the two forces.

Result. $\mathrm{Sin}^{-1}\frac{5}{16}(10-2\sqrt{5})^{\frac{1}{2}}$.

6. The resultant of two forces P, Q, acting at an angle θ, is equal to $(2m+1)\sqrt{(P^2+Q^2)}$; when they act at an angle $\frac{1}{2}\pi-\theta$, it is equal to $(2m-1)\sqrt{(P^2+Q^2)}$; shew that $\tan\theta=\frac{m-1}{m+1}$.

7. Two forces F and F' acting in the diagonals of a parallelogram keep it at rest in such a position that one of its edges is horizontal, shew that

$$F \sec \alpha' = F' \sec \alpha = W \operatorname{cosec} (\alpha + \alpha'),$$

where W is the weight of the parallelogram, α and α' the angles between its diagonals and the horizontal side.

8. If a particle be placed on a sphere, and be acted on by three forces represented in magnitude and direction by three chords mutually at right angles drawn through the particle, it will remain at rest.

9. Three forces P, Q, R acting on a point and keeping it at rest are represented by straight lines drawn from that point. If P be given in magnitude and direction, and Q in magnitude only, find the locus of the extremity of the line which represents the third force R.

Result. A sphere.

10. A circle whose plane is vertical has a centre of constant repulsive force at one extremity of the horizontal diameter; find the position of equilibrium of a particle within the circle, the repulsive force being equal to the weight of the particle.

Result. The straight line joining the particle with the centre of the circle makes an angle of 60° with the horizon.

11. A particle is placed on a smooth square table whose side is a at distances c_1, c_2, c_3, c_4 from the corners, and to it are attached strings passing over smooth pullies at the corners and supporting weights P_1, P_2, P_3, P_4; shew that if there is equilibrium,

$$\left(\frac{P_1}{c_1} + \frac{P_2}{c_2} + \frac{P_3}{c_3} + \frac{P_4}{c_4}\right)^2 \frac{c_1^2}{a^2} = \left(\frac{P_2}{c_2} + \frac{P_3}{c_3}\right)^2 + \left(\frac{P_3}{c_3} + \frac{P_4}{c_4}\right)^2.$$

Shew also that

$$\left(\frac{P_1}{c_1} + \frac{P_2}{c_2} + \frac{P_3}{c_3} + \frac{P_4}{c_4}\right) \frac{c_1^2 - c_3^2}{a^2} = 2\left(\frac{P_3}{c_3} - \frac{P_1}{c_1}\right).$$

12. Two small rings slide on the arc of a smooth vertical circle; a string passes through both rings, and has three equal weights attached to it, one at each end and one between the rings; find the position of the rings when they are in equilibrium. The rings are supposed without weight.

Result. Each of the rings must be 30° distant from the highest point of the circle.

13. The extremities of a string without weight are fastened to two equal heavy rings which slide on smooth fixed rods in the same vertical plane and equally inclined to the vertical; and to the middle point of the string a weight is fastened equal to twice the weight of each ring; find the position of equilibrium and the tension of the string.

If the point to which the weight is fastened be not the middle point of the string, shew that in the position of equilibrium the tensions of its two portions will be equal.

14. A light cord with one end attached to a fixed point passes over a pully in the same horizontal line with the fixed point and supports a weight hanging freely at its other end. A heavy ring being fastened to the cord in different places between the fixed point and the pully, it is required to find the locus of its positions of equilibrium. If the weight of the ring be small compared with the other weight, the locus will be approximately a parabola.

15. If two forces acting along chords of a circle are inversely proportional to the lengths of the chords, their resultant will pass through one or other of the points of intersection of straight lines drawn through the extremities of the chords.

16. A particle rests on an ellipse acted on by forces λx^n, μy^n, parallel to the axes of x and y respectively; find its position of equilibrium. Explain the case in which $n = 1$.

17. A particle is placed on the outer surface of a smooth fixed sphere and is acted on by a fixed centre of force lying vertically above the centre of the sphere, at a distance c from it and attracting directly as the distance. Shew that the

particle will rest on any part of the sphere if the weight of the particle equals the attraction on it by the fixed centre of force when at a distance c from it.

18. A particle is placed on the surface of an ellipsoid in the centre of which is resident an attractive force: determine the direction in which the particle will begin to move.

19. Find the point on the surface $\frac{x^3}{a^3}+\frac{y^3}{b^3}+\frac{z^3}{c^3}=1$, where a particle attracted by a force to the origin will rest in equilibrium.

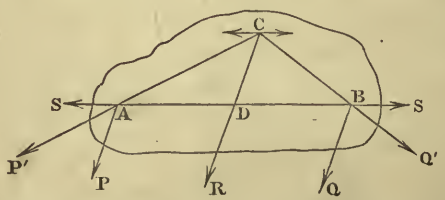
20. $ABCD$ is a quadrilateral inscribed in a circle, and forces inversely proportional to AB, BC, AD, DC act along the sides in the directions indicated by the letters: shew that their resultant acts along the straight line joining the intersection of the diagonals with the intersection of the tangents to the circle at B, D.

CHAPTER III.

RESULTANT OF TWO PARALLEL FORCES. COUPLES.

37. *To find the magnitude and the direction of the resultant of two parallel forces acting on a rigid body.*

Let P and Q be the forces; A and B their points of ap-



plication: let P and Q act in the *same* direction. The effect of the forces will not be altered if we apply two forces equal in magnitude and acting in opposite directions along the straight line AB. Let S denote each of these forces, and suppose one to act at A and the other at B.

Then P and S acting at A are equivalent to some force P' acting in some direction AP' inclined to AP (Art. 17); and Q and S acting at B are equivalent to some force Q' acting in some direction BQ' inclined to BQ.

Produce $P'A$, $Q'B$ to cut each other at C, and draw CD parallel to AP, meeting AB at D; suppose C rigidly connected with AB.

Transfer P' and Q' to C (Art. 11), and resolve them along CD and a straight line parallel to AB; the latter parts will each be equal to S but act in opposite directions, and the sum of the former is $P+Q$. Hence R, the resultant of P and Q, $=P+Q$ and acts parallel to P and Q in the straight line CD. We shall now determine the point where this straight line cuts AB.

The sides of the triangle ACD are parallel to the directions of the forces P, S, P'; therefore by Art. 19

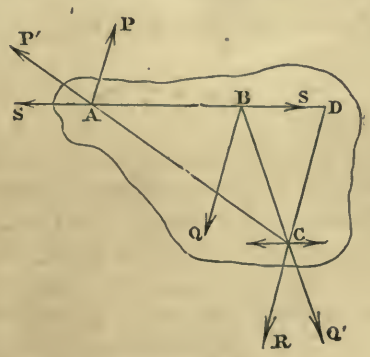
$$\frac{P}{S} = \frac{CD}{DA}, \text{ and similarly } \frac{S}{Q} = \frac{DB}{CD};$$

therefore $\frac{P}{Q} = \frac{DB}{DA} = \frac{a-x}{x}$, if $AB = a$ and $AD = x$;

therefore
$$\frac{x}{a} = \frac{Q}{P+Q};$$

this determines the point D through which the direction of the resultant passes. It will be observed that AB is divided at D into segments which are inversely as the forces at A and B respectively.

If the force P act in a direction *opposite* to that of Q



a similar process will lead us to

$$R = Q - P, \text{ and } \frac{x}{a} = \frac{Q}{Q-P};$$

which may be derived from the formulæ of the preceding case by changing P into $-P$.

It will be observed that AB produced is divided at D into segments which are inversely as the forces at A and B respectively.

38. The point D possesses this remarkable property: that however P and Q are turned about their points of application A and B, their directions remaining parallel, D determined as above remains fixed. This point is in consequence called the *centre* of the parallel forces P and Q.

39. If $P=Q$ in the second case of Art. 37, then $R=0$ and $x=\infty$, a result perfectly nugatory. It shews us that the method fails by which we have attempted to compound two equal and opposite parallel forces. In fact the addition of the two forces S still gives, in this case, two equal forces parallel and opposite in their directions.

Such a system of forces is called a *Couple.*

We shall investigate the laws of the composition and resolution of couples, since we shall afterwards have occasion to employ them extensively in the composition and resolution of forces of every description acting on a rigid body.

40. A *couple* consists of two equal forces acting in parallel and opposite directions.

The *arm* of a couple is the perpendicular distance between the lines of action of its forces.

The *moment* of a couple is the product of either of its forces into the perpendicular distance between them.

The *axis* of a couple is a straight line perpendicular to the plane of the couple and proportional in length to the moment.

Two couples in the same plane may differ with respect to direction. For suppose the middle point of the arm of a couple to be fixed, and the arm to move in the direction in which the two forces of the couple tend to urge it; there are two different directions in which the arm may rotate. Suppose a perpendicular drawn to the plane of the couple through the middle point of its arm, so that when an observer is placed along this straight line with his feet against the plane, the rotation which the forces give to the arm appears to take place from *left* to *right;* the perpendicular so drawn we shall take for the *axis* of the couple.

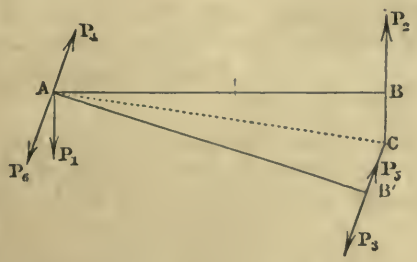
[It is universally agreed that if Ox, Oy, Oz are the positive directions along three rectangular axes, a couple tending to

turn from Ox to Oy shall have its axis in the direction Oz; but opposite conventions are in use as to the direction in which Oz shall be drawn. The convention adopted in Thomson and Tait's *Natural Philosophy*, in Tait's *Quaternions*, and in Ball's *Theory of Screws*, is that which is suggested by the motion of an ordinary or right-handed screw. The convention uniformly adopted in the present work is that which is suggested by the motion of a left-handed screw.]

41. We shall now give three propositions shewing that the effect of a couple is not altered when certain changes are made with respect to the couple. It is to be supposed in all these propositions that a rigid body is in equilibrium under the action of certain forces, including an assigned couple; and it is shewn that then the equilibrium will not be disturbed by the specified changes with respect to the couple.

42. *The effect of a couple is not altered if its arm be turned through any angle about one extremity in the plane of the couple.*

Let the plane of the paper be the plane of the couple, AB the arm, and AB' its new position; the forces P_1, P_2 are equal



and act on the arm AB. At B' and A let the equal and opposite forces $P_3 P_5$, $P_4 P_6$, each equal to P_1 or P_2 be applied, acting at right angles to AB'; this will not affect the action of P_1 and P_2.

Let BP_2, $B'P_3$ meet at C; join AC; AC manifestly bisects the angle BAB'.

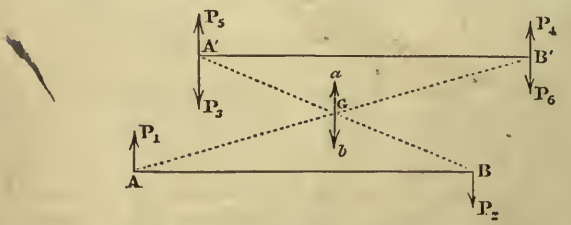
Now P_2 and P_3 are equivalent to some force in the direction CA, and P_1 and P_4 are equivalent to an *equal* force in the direc-

tion AC. Therefore P_1, P_2, P_3, P_4 are in equilibrium with each other; therefore the remaining forces P_5, P_6 acting at B', A respectively produce the same effect as P_2, P_1 acting at B, A respectively. Hence the proposition is true.

We may now turn the arm of the couple through any angle about B'; and by proceeding in this way we may transfer the couple to any position in its own plane.

43. *The effect of a couple is not altered if we transfer the couple to any plane parallel to its own, the arm remaining parallel to itself.*

Let AB be the arm, $A'B'$ its new position parallel to AB.



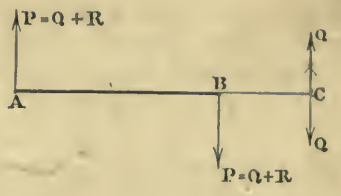
Join AB', $A'B$ bisecting each other at G. At A', B' apply two equal and opposite forces each $= P_1$ or P_2 and parallel to them; and let these forces be P_3, P_4, P_5, P_6; this will not alter the effect of the couple.

But P_1 and P_4 are equivalent to $2P_1$ acting at G in the direction Ga parallel to the direction of P_1, and P_2 and P_3 are equivalent to $2P_1$ acting at G in the opposite direction Gb.

Hence P_1, P_2, P_3, P_4 are in equilibrium with each other; therefore the remaining forces P_5 and P_6 acting at A' and B' respectively produce the same effect as P_1 and P_2 acting at A and B respectively. Hence the proposition is true.

44. *The effect of a couple is not altered if we replace it by another couple of which the moment is the same; the plane remaining the same and the arms being in the same straight line and having a common extremity.*

Let AB be the arm; let P, P be the forces, and suppose $P = Q + R$; let $AB = a$, and let the new arm $AC = b$; at C apply two opposite forces each $= Q$ and parallel to P; this will not alter the effect of the couple.



Now R at A and Q at C will balance $Q + R$ at B,

if $AB : BC :: Q : R$, (Art. 37),

or if $AB : AC :: Q : Q + R$,

that is, if $Q \cdot b = P \cdot a$;

we have then remaining the couple Q, Q acting on the arm AC. Hence the couple P, P acting on AB may be replaced by the couple Q, Q acting on AC, if $Q \cdot b = P \cdot a$, that is, if their moments are the same.

45. From the last three Articles it appears that, without altering the effect of a couple, we may change it into another of equal moment, and transfer it to any position, either in its own plane or in a plane parallel to its own. The couple must remain unchanged so far as concerns the direction of the rotation which its forces would tend to give the arm, supposing its middle point fixed as in Art. 40. In other words, the straight line which we have called the axis, measured as indicated in that Article, must always remain on the *same side* of the plane of the couple.

46. We may infer from Art. 44 that couples may be measured by their moments. Let there be two couples, one in which each force $= P$, and one in which each force $= Q$, the arms of the couples being *equal;* these couples will be in the ratio of P to Q. For suppose, for example, that P is to Q as 3 to 5; then each of the forces P may be divided into 3 equal forces and each of the forces Q into 5 such equal forces. Then the couple of which each force is P may be considered as the sum of 3 equal couples of the same kind, and the couple of which each force is Q as the sum of 5 such equal couples. The effects of the couples will therefore be as 3 to 5. Next,

suppose the arms of the couples *unequal*, and denote them by p and q respectively. The couple which has each of its forces $= Q$ and its arm $= q$ is equivalent to a couple having each of its forces $= \frac{Qq}{p}$ and its arm $= p$, by Art. 44. The couples are therefore by the first case in the ratio of P to $\frac{Qq}{p}$, that is of Pp to Qq.

With respect to the effect of a couple, we may observe that it is shewn in works on rigid dynamics that if a couple act on a free rigid body it will set the body in rotation about an axis passing through a certain point in the body called its centre of gravity, *but not necessarily perpendicular to the plane of the couple*, the direction of the initial axis being influenced by the distribution of the mass of the body.

47. *To find the resultant of any number of couples acting on a body, the planes of the couples being parallel to each other.*

First, suppose all the couples transferred to the same plane (Art. 43); next, let them be all transferred so as to have their arms in the same straight line, and one extremity common (Art. 42); and lastly, let them be replaced by other couples having the same arm (Art. 44).

Thus if $P, Q, R, \ldots\ldots$ be the forces, and

$a, b, c, \ldots\ldots$ be their arms,

we shall have them replaced by the following forces (supposing α the common arm),

$$P.\frac{a}{\alpha}, \quad Q.\frac{b}{\alpha}, \quad R.\frac{c}{\alpha}, \ldots\ldots \text{ acting on the arm } \alpha.$$

Hence their resultant will be a couple of which each force equals

$$P.\frac{a}{\alpha} + Q.\frac{b}{\alpha} + R.\frac{c}{\alpha} + \ldots\ldots,$$

and the arm $= \alpha$,

or of which the moment equals

$$P.a + Q.b + R.c + \ldots\ldots$$

Hence the moment of the resultant couple is equal to the sum of the moments of the original couples.

If one of the couples, as Q, Q, act in a direction *opposite* to the couple P, P, then the force at each extremity of the arm of the resultant couple will be

$$P.\frac{a}{\alpha} - Q.\frac{b}{\alpha} + R.\frac{c}{\alpha} + \ldots\ldots,$$

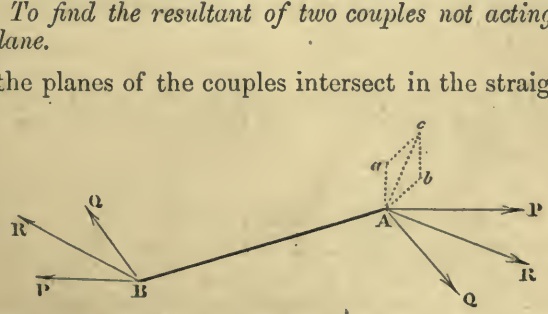
and the moment of the resultant couple will be

$$P.a - Q.b + R.c + \ldots\ldots,$$

or the algebraical sum of the moments of the original couples; the moments of those couples which tend in the direction opposite to the couple P, P being reckoned negative.

48. *To find the resultant of two couples not acting in the same plane.*

Let the planes of the couples intersect in the straight line



AB, which is perpendicular to the plane of the paper, and let the couples be referred to the common arm AB, and let their forces thus altered be P and Q.

In the plane of the paper draw Aa, Ab at right angles to the planes of the couples P, P and Q, Q; and equal in length to their *axes*.

Let R be the resultant of the forces P and Q at A, acting in the direction AR; and of P and Q at B, acting in the direction BR.

Since AP, AQ are parallel to BP, BQ respectively, therefore AR is parallel to BR.

Hence the two couples are equivalent to the single couple R, R acting on the arm AB.

Draw Ac perpendicular to the plane of R, R, and in the same proportion to Aa, Ab that the moment of the couple R, R is to those of P, P and Q, Q respectively. Then Ac is the *axis* of R, R. Now the three straight lines Aa, Ac, Ab make the same angles with each other that AP, AR, AQ make with each other; also they are in the same proportion in which $AB.P$, $AB.R$, $AB.Q$ are; that is in which P, R, Q are.

But R is the resultant of P and Q; therefore Ac is the diagonal of the parallelogram on Aa, Ab (see Art. 17).

Hence if two straight lines, having a common extremity, represent the axes of two couples, that diagonal of the parallelogram described on these straight lines as adjacent sides which passes through their common extremity represents the axis of the resultant couple.

49. *To find the magnitude and position of the couple which is the resultant of three couples which act in planes at right angles to each other.*

Let AB, AC, AD be the axes of the given couples (see fig. to Art. 24). Complete the parallelogram CB, and draw AE the diagonal. Then AE is the axis of the couple which is the resultant of the two couples of which the axes are AB, AC. Complete the parallelogram DE, and draw AF the diagonal. Then AF is the axis of the couple which is the resultant of the couples of which the axes are AE, AD, or of those of which the axes are AB, AC, AD.

Now $$AF^2 = AE^2 + AD^2 = AB^2 + AC^2 + AD^2.$$

Let G be the moment of the resultant couple; L, M, N those of the given couples;

therefore $$G^2 = L^2 + M^2 + N^2;$$

and if λ, μ, ν be the angles the axis of the resultant makes

with those of the components,

$$\cos\lambda = \frac{AB}{AF} = \frac{L}{G}; \quad \cos\mu = \frac{M}{G}; \quad \cos\nu = \frac{N}{G}.$$

50. Hence conversely any couple may be replaced by three couples acting in planes at right angles to each other; their moments being $G\cos\lambda$, $G\cos\mu$, $G\cos\nu$; where G is the moment of the given couple, and λ, μ, ν the angles its axis makes with the axes of the three couples.

Thus *couples* follow, as to their composition and resolution, the same laws as *forces acting at a point*, the *axis* of the couple corresponding to the *direction* of the force and the *moment* of the couple to the *magnitude* of the force. Hence for example, by Art. 29, the resolved part of a resultant couple in any direction is equal to the sum of the resolved parts of the component couples in the same direction.

51. In Chapter II. we treated of forces acting *at a point;* in the present Chapter we have passed to the consideration of forces which do *not* act at the same point, and this subject will be further developed in the following Chapters. Mistakes are frequently made by students in consequence of neglecting the difference between the case in which forces are given acting at the *same point*, and the case in which the forces are *not* so given.

It may be observed that by the aid of what has already been explained, it is theoretically possible to discuss any problem respecting forces which all act in the same plane; we have only to make repeated use of the principles of transference and composition. For take any two of the forces, produce the directions of the forces to meet, and suppose the two forces to act at the common point; then replace the two forces by their resultant. Next this resultant may in like manner be compounded with any third force; and so on. Thus we shall always be able to reduce any forces which act in the same plane either to a single force, or to a couple. The next Chapter will shew us how to proceed in a more systematic manner; but, as we have said, the explanations already given are theoretically sufficient.

For example. Forces are represented in magnitude and line of action by the sides AB, BC, and CA of a triangle: find their resultant.

The two forces AB and BC may be supposed to act at B; their resultant will be a force through B, parallel to AC, and denoted in magnitude by AC. Then this force and the force represented by CA constitute a couple; the moment of the couple will be represented by the product of CA into the perpendicular from B on CA; that is by twice the area of the triangle ABC.

Again. Forces are represented in magnitude and line of action by the sides AB, BC, and AC of a triangle: find their resultant.

Here, as before, the two forces AB and BC are equivalent to a force through B parallel to AC; thus the final resultant is a force parallel to AC, passing through the middle point of AB, and denoted in magnitude by twice AC.

EXAMPLES.

1. Forces $1, -2, -3, 4$ act on a rod at right angles to it, and at equal distances in the same plane: find their resultant.

2. Forces $P, -3P, -5P, 7P$ act on a rod at distances from one end proportional to their respective magnitudes; the forces are parallel and in the same plane: find their resultant.

3. $ABCD$ is a quadrilateral; forces are represented in magnitude and line of action by AB, BC, and CD: find their resultant.

4. $ABCD$ is a quadrilateral; forces are represented in magnitude and line of action by AB, BC, and DC: find their resultant.

5. $ABCD$ and $A'B'C'D'$ are parallel faces of a parallelepiped, the edges AA', BB', CC', DD', being parallel to each other; if forces represented by the edges BA, BC, $D'C'$, $D'A'$ act respectively along these edges, shew that they are equivalent to a couple acting in the plane passing through BB' and DD', and represented in moment by the area of the section of the parallelepiped made by that plane.

CHAPTER III. A.

GRAPHICAL STATICS.

MUCH attention has been paid in recent years to the solution of statical problems by graphical construction, especially in cases where it is desired to determine the internal forces produced in the bars of a jointed frame by the action of external forces at the joints. The process of construction in these cases consists of repeated applications of the "Triangle" or "Polygon of Forces", and every line that represents an internal force is usually a common side of two triangles or polygons in which it represents respectively the two opposite pulls or pushes which the bar exerts at its two ends. Every such line is accordingly taken in opposite directions in the two triangles or polygons. When the weight of a bar is to be taken into account, it must be resolved into two parallel forces acting at the two ends of the bar, (or more exactly, at the joints,) and these must be included among the external forces.

The internal forces which we here consider are directed along the length of each bar, and may be either pushes or pulls. When they are pulls, the bars act like strings; and for greater clearness of conception we shall begin with a case in which all the bars are actually strings, namely the case of a triangle of string kept tight by three strings pulling outwards at its three corners.

Let A, B, C (fig. 1) be the sides of the triangle of string, and P, Q, R the forces with which the corners are pulled outwards, which we shall call the external forces. Since the system is in equilibrium, the lines of action of the three external forces must meet in a point in the plane of the triangle; and since the strings which form the sides of the triangle can only pull and not push, this point must lie within the triangle.

Draw a triangle pqr having its sides parallel to P, Q, R respectively; they will represent P, Q, R in magnitude, and it is convenient to mark them with arrows to shew in which direction they are to be taken. In all our diagrams of forces, lines marked with arrows will represent *external* forces.

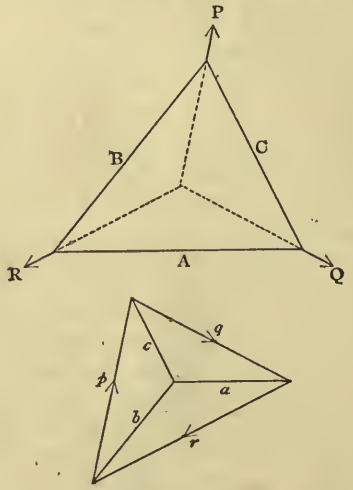


Fig. 1.

From the point (pq) where the two sides p, q meet draw a line c parallel to C, and from the point (pr) draw b parallel to B; then pcb is the triangle of forces for the three balancing forces which act at the point of application of P.

Join the point (bc) to the point (qr) by a line a. Then in the triangle cqa since the sides c, q represent two of the three forces which make equilibrium at the corner (AC), the third side a must represent the third force, namely the pull in the side A. Hence the six lines p, q, r, a, b, c in our force-diagram represent the three external forces P, Q, R, and the three pairs of internal forces in the bars A, B, C. It is noteworthy that we have here a mechanical proof of the geometrical proposition that the line joining (bc) to (qr) is parallel to A.

The three sides of the triangle rba represent the three forces which make equilibrium at the corner (AB), and we

have in all in our force-diagram four triangles of forces, one for each of the four points in which lines meet in the upper figure. The correspondence is indicated by the lettering, thus the triangle *pqr* corresponds to the point (*PQR*) where the three lines *P*, *Q*, *R* meet, the triangle *pcb* to the point (*PCB*) and so on. In each of the four triangles of forces the direction in which we must travel round the perimeter to shew the directions of the forces is the direction in which the hands of a watch revolve. This is clear from the directions of the three arrows, for one at least of the sides thus marked occurs in each triangle. The sides not so marked represent internal forces and each of them is taken in opposite directions in the two triangles to which it belongs.

The geometrical relation of the first figure to the second is the same as that of the second to the first; lines which meet in a point in either figure are parallel to the sides of

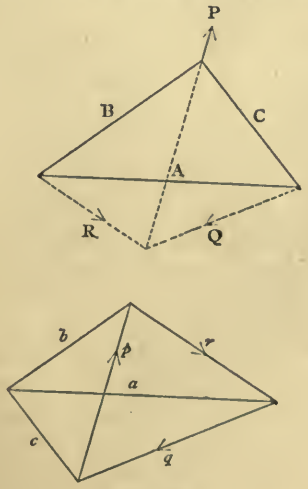


Fig. 2.

a triangle in the other. Hence if we suppose a string-triangle *pqr* to be pulled outwards at its corners by forces acting along the lines *a*, *b*, *c*, the six lines in the diagram *ABCPQR* will represent these three forces and the three pairs of resulting pulls in the sides.

When two diagrams, as here, are so related that each will serve as the force-diagram for the other, they are called *reciprocal diagrams*. The properties of such diagrams have been investigated by Prof. Clerk Maxwell, (*Phil. Mag.* XXVII 1864, p. 250,) and a somewhat simpler discussion of the subject will be found in the *Statique Graphique* of M. Maurice Lévy.

If we now suppose the bars A, B, C to be stiff, equilibrium will still exist when the forces P, Q, R are reversed, and the push in each bar will now be equal to the former pull. The force-diagram is unchanged except that the arrows must all be reversed.

When the bars are stiff, the point of meeting of P, Q, R may be without the frame ABC. One such case is represented in fig. 2, the diagram of forces being constructed by the same process as in the first case. The internal forces consist of a push in A and pulls in B and C.

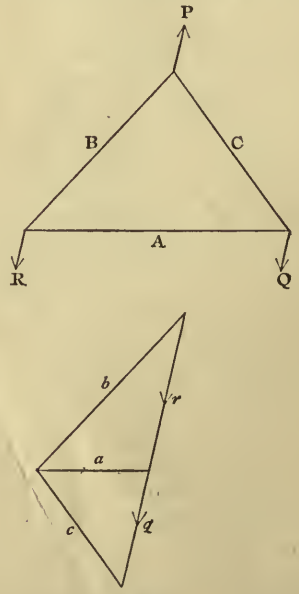


Fig. 3.

When the forces P, Q, R are parallel, as in fig. 3, the triangle

pqr shrinks into a straight line or rather into a pair of coincident straight lines, one of the three sides (in the case here represented, p,) being the sum of the other two. The two arrows which we have inserted indicate the directions of q and r which are sides of the triangles qca, rab. The line p, which is not separately shewn, as it coincides with q and r, is to be taken in the opposite direction, and is a side of the triangle pbc.

We will next take the case of a plane polygonal frame of any number of sides given in position as well as in length. The lines of action of the external forces at all of its corners except one may be chosen at pleasure. There will then be sufficient data for determining the line of action of the remaining external force and the ratios of all the forces both external and internal.

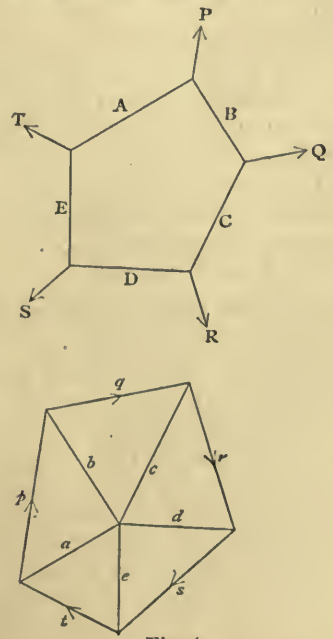


Fig. 4.

Let $ABCDE$ (fig. 4) be the given frame, P, Q, R, S the forces whose lines of action are arbitrarily assumed, T the

force whose line of action is to be found. Draw a triangle of forces *pba* for the external force P and the two internal forces which equilibrate it at the point (AB). From the corner (pb) draw q to represent the direction of the force Q, and complete the triangle of forces *bqc* for the three forces which make equilibrium at the point (BC). In like manner on c construct the triangle *crd* to represent the forces at (CD), and on d the triangle *dse* for the forces at (DE). We have thus constructed an open polygon *pqrs* with lines a, b, c, d, e radiating to its corners from a single point; and the nine lines $p, q, r, s, a, b, c, d, e$ represent the external forces P, Q, R, S, and the five pairs of internal forces in A, B, C, D, E. Close the polygon by the side t drawn from (se) to (ap). This side will represent the required force T, for by the triangle of forces *tae* it balances the internal forces in A and E.

Since the bar B is in equilibrium under the action of P, Q, and the internal forces in A, C, the resultant of the two former must have the same line of action as the resultant of the two latter; hence the resultant of P and Q passes through the meeting-point of A, C. In like manner the resultant of P, Q and R must pass through the meeting-point of A, D. Each of these resultants is represented by one of the diagonals of the polygon *pqrst*. The resultant of P, Q, R and S is the force T reversed.

By inverting the above construction we can find the resultant of any given forces in one plane.

Let P, Q, R, S be the forces. Draw the open polygon *pqrs* to represent them, and close it by the side t, as above. Then the resultant will be represented by t taken in the reverse direction. To find its line of action, draw lines a, b, c, d, e from any point to the corners of the polygon *pqrst*. Then, b being the line drawn to the point (pq), draw any line B parallel to it terminated by the lines of action of P and Q. In like manner draw C parallel to c and D parallel to d. Complete the polygon $ABCDE$ by drawing A and E parallel to a and e until they meet. Their point of meeting will be a point on the line of action of the required resultant, since by the foregoing proof a force T applied at this point will balance P, Q, R, S.

When any number of the forces P, Q, R... applied at consecutive joints of a frame are parallel, as in fig. 5, the corresponding sides p, q, r... of the force-diagram are in one straight line.

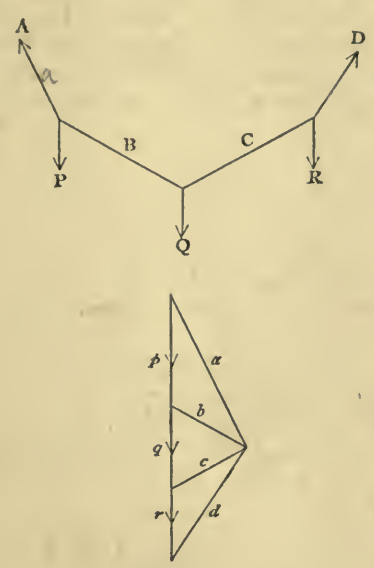


Fig. 5.

In our next example, we shall employ a different mode of lettering which is often convenient. Each letter will denote in one diagram a space and in the other a point. Each line will be denoted by two letters, which in the first diagram will be the names of the spaces which the line separates, and in the second diagram will be the names of the points which the corresponding line joins.

The frame represented in the first part of fig. 6 consists of seven bars, and external forces P, Q, R are applied at three of its joints, their lines of action meeting in a point. The three triangles which compose the frame are distinguished by the letters a, b, c, and the two bars which separate them will be called ab, bc. The space outside the frame is divided into three parts by the lines of action of the three external forces drawn from their points of application in the forward

directions only. These parts are designated *d*, *e*, *f*, and the lines of action of *P*, *Q*, *R* will be called *df*, *ef*, *de*. The point of application of *P* is the meeting-point of the three lines *df*, *da*, *af*, or of the three spaces *a*, *d*, *f*, and the force-diagram may be commenced by drawing the triangle *adf* to represent the three forces which balance each other at this point. We next construct the triangle *afb* to represent the three internal forces which balance at the point *afb*;

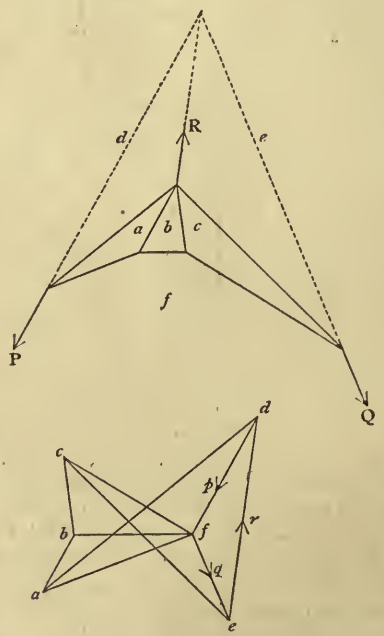


Fig. 6.

then the triangle *bfc* for the forces at the point *bfc*; then the triangle *cfe* for the point *cfe*, one of the three forces here being the external force *Q*. If we now join the two points *e*, *d* the joining line will represent *R*, since *R* balances *P* and *Q* which are represented by the two sides *df*, *fe*. At the point of application of *R*, which is the point of junction of the spaces *a*, *b*, *c*, *e*, *d*, there are five forces in equilibrium,

and they are represented by the sides of the polygon *abced* in the force-diagram.

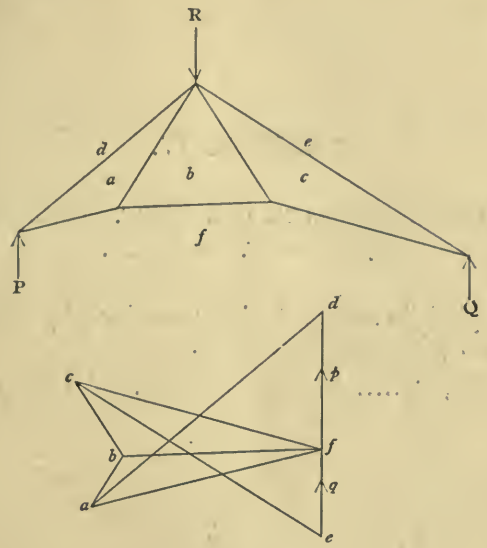


Fig. 7.

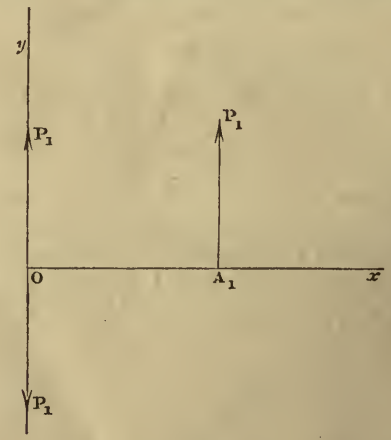
When the three external forces P, Q, R are parallel, as in fig. 7, the triangle *dfe* shrinks into two coincident straight lines, and the whole line *de*, representing the force R, is to be taken in the opposite direction to the two arrows which apply to the parts *ef*, *fd* representing Q and P.

CHAPTER IV.

RESULTANT OF FORCES IN ONE PLANE. CONDITIONS OF EQUILIBRIUM. MOMENTS.

52. *To find the resultant of any number of parallel forces acting on a rigid body in one plane.*

Let $P_1, P_2, P_3 \ldots\ldots$ denote the forces. Take any point in

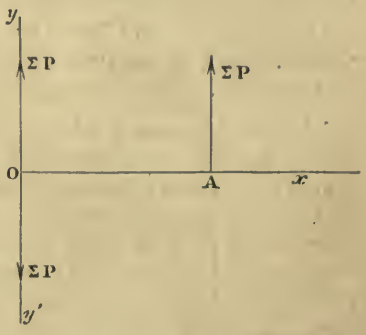


the plane of the forces as origin and draw rectangular axes Ox, Oy, the latter parallel to the forces. Let A_1 be the point where Ox meets the direction of P_1, and let $OA_1 = x_1$.

Apply at O two forces each equal and parallel to P_1, in opposite directions. Thus the force P_1 is replaced by P_1 at O along Oy, and a couple of which the moment is $P_1 . OA_1$, that is $P_1 . x_1$. Transform the other forces in a similar manner,

using a similar notation, and the whole system will be reduced to a force $P_1 + P_2 + P_3 + \ldots\ldots$ or ΣP along Oy, and a couple $P_1x_1 + P_2x_2 + P_3x_3 + \ldots\ldots$ or ΣPx in the plane of the forces and tending to turn the body from the axis of x to the axis of y.

53. When the sum of the forces vanishes in Art. 52, the forces reduce to a couple. When ΣP is not zero, the forces can be reduced to a single force. For if $\Sigma Px = 0$, then ΣP acting at O is *the single* resultant. If ΣPx be not $= 0$, let the couple be transformed to one in which each of the forces is equal to ΣP, and consequently, by Art. 44, the arm is $\frac{\Sigma Px}{\Sigma P}$. Let ΣP acting at A



and ΣP acting along Oy' form this couple. The latter force is destroyed by the force ΣP along Oy. Hence the single resultant is ΣP acting at A, that is, at a point the distance of which from O is $\frac{\Sigma Px}{\Sigma P}$.

54. *To find the conditions of equilibrium of a system of parallel forces acting on a rigid body in one plane.*

The system can be reduced to a single force acting at an arbitrary origin and a couple. If the single force alone vanish equilibrium is impossible, because there remains an unbalanced couple. If the couple alone vanish equilibrium is impossible, because there remains an unbalanced force. If neither the force nor the couple vanish, they can be reduced to a single force equal to the given force, as shewn in Art. 53. Hence, for equilibrium it is necessary that both the force and the couple should vanish; that is

$$\Sigma P = 0 \text{ and } \Sigma Px = 0.$$

55. The product of a force into the perpendicular drawn upon its line of action from any point, is called the moment

of the force with respect to that point, or round that point. Hence the conditions of equilibrium which have just been obtained may be thus enunciated:

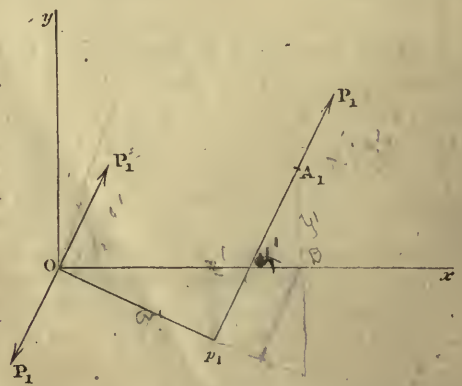
A system of parallel forces acting on a rigid body in one plane will be in equilibrium if the sum of the forces vanishes, and the sum of the moments of the forces round an origin in the plane also vanishes.

Conversely, if the forces are in equilibrium their sum must vanish, and also the sum of their moments round any origin in the plane.

The word *sum* must be understood *algebraically.* Forces which act in one direction being considered *positive,* those which act in the opposite direction must be considered *negative.* Also moments being considered *positive* when the corresponding couples tend to turn the body in *one* direction, they must be considered *negative* when the corresponding couples tend to turn the body in the *opposite* direction.

56. *To find the resultant of any number of forces which act on a rigid body in one plane.*

Let the system be referred to any rectangular axes *Ox, Oy* in the plane of the forces.



Let P_1, P_2, P_3, ... denote the forces; α_1, α_2, α_3, ... the angles which their directions make with the axis of x;

let x_1, y_1 be the co-ordinates of the point of application of P_1; let x_2, y_2 be those of the point of application of P_2, and so on.

Let A_1 be the point of application of P_1. At O suppose two forces applied in opposite directions each equal and parallel to P_1. Draw Op_1 perpendicular to P_1A_1.

Then P_1 acting at A_1 is equivalent to P_1 acting at O and a couple of which Op_1 is the arm and each force is P_1, which tends to turn the body from the axis of x to that of y. Now

$$Op_1 = x_1 \sin \alpha_1 - y_1 \cos \alpha_1.$$

Hence the moment of the couple is

$$P_1 (x_1 \sin \alpha_1 - y_1 \cos \alpha_1).$$

The other forces may be similarly replaced. Hence the system is equivalent to the forces

$$P_1, P_2, P_3, \ldots\ldots \text{ acting at } O,$$

in directions parallel to those of the original forces; and the couples of which the moments are

$$P_1 (x_1 \sin \alpha_1 - y_1 \cos \alpha_1),$$

$$P_2 (x_2 \sin \alpha_2 - y_2 \cos \alpha_2),$$

$$P_3 (x_3 \sin \alpha_3 - y_3 \cos \alpha_3),$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

acting in the plane of the forces. It will be found that any one of the above expressions for the moments of the couples is positive or negative, according as that couple tends to turn the body from the axis of x towards that of y, or in the contrary direction.

Let R be the resultant of the forces acting at O, let a be the angle which R makes with the axis of x, and G the moment of the resultant couple; then (by Art. 22)

$$R \cos a = \Sigma P \cos \alpha; \quad R \sin a = \Sigma P \sin \alpha;$$

and (by Art. 47)

$$G = \Sigma P (x \sin \alpha - y \cos \alpha).$$

If $P_1 \cos \alpha_1 = X_1$ and $P_1 \sin \alpha_1 = Y_1$, and a similar notation be used for the other forces, the above equations may be written

$$R^2 = (\Sigma X)^2 + (\Sigma Y)^2; \quad \tan a = \frac{\Sigma Y}{\Sigma X};$$

and

$$G = \Sigma (Yx - Xy).$$

57. *To find the conditions for the equilibrium of a system of forces acting on a rigid body in one plane.*

Any system of forces acting in one plane may be reduced to a single force R, and a couple whose moment is G. If R alone vanish equilibrium is impossible, because there remains an unbalanced couple G; if G alone vanish equilibrium is impossible, because there remains an unbalanced force. If neither R nor G vanish they may be reduced as in Art. 53 to a single force equal to R. Hence, for equilibrium we must have $R = 0$ and $G = 0$. Also $R = 0$ requires that $\Sigma X = 0$ and $\Sigma Y = 0$.

Since G is equal to the sum of the moments of the forces with respect to O, we may enunciate the result thus: *A system of forces acting in one plane on a rigid body will be in equilibrium if the sums of the resolved parts of the forces parallel to two rectangular axes in the plane vanish, and the sum of the moments round an origin in the plane also vanishes.*

Conversely, if the forces are in equilibrium the sum of the resolved parts of the forces parallel to any straight line will vanish, and also the sum of the moments of the forces round any origin.

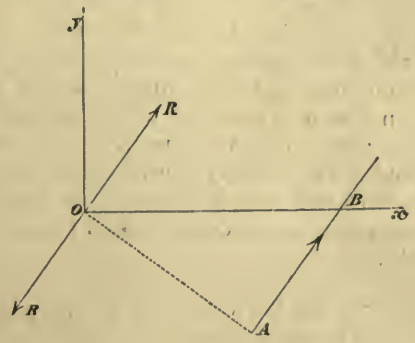
58. *If three forces acting in one plane maintain a rigid body in equilibrium their lines of action either all meet at a point or are all parallel.*

For suppose two of the lines of action to meet at a point, and take this point for the origin; then the moment of each of these two forces vanishes, and the equation $G = 0$ requires that the moment of the third force should vanish, that is, the third force must also pass through the origin. Hence, if any two of the forces meet, the third must pass through their point of intersection, which proves the proposition. This pro-

position may also be established without referring to Art. 57. For if two of the forces meet at a point, they may be supposed both to act at that point and may be replaced by their resultant acting at the same point; this resultant and the third force must keep the body on which they act in equilibrium, and must therefore be equal and opposite; that is, the third force must pass through the point of intersection of the first two.

59. If $R = 0$ in Art. 56, the forces reduce to a couple: if R be not $= 0$, the forces can be reduced to a single force; and the equation to its line of action may be found.

For if the couple $G = 0$, the resultant force is R acting at the origin. If the couple G be not $= 0$, let it be transformed into one having each of its forces $= R$ and its arm consequently $= \frac{G}{R}$ (Art. 44). Let this couple be turned in its own plane, until one of its forces acts at the origin exactly opposite to the force R, which by hypothesis acts at the origin. Hence these forces destroy each other and we have left R acting at the extremity of the arm OA, in a direction inclined to the axis of x at an angle a, found by the equation $\tan a = \frac{\Sigma Y}{\Sigma X}$ (Art. 56). If this force meet the axis of x at B, we have



$$OB = OA \operatorname{cosec} a = \frac{G}{R} \cdot \frac{R}{\Sigma Y} = \frac{G}{\Sigma Y},$$

and the equation to the line of action of the single resultant is

$$y' = \frac{\Sigma Y}{\Sigma X}\left(x' - \frac{G}{\Sigma Y}\right);$$

or,
$$x'\Sigma Y - y'\Sigma X = \Sigma\,(Yx - Xy),$$

x', y' being the variable co-ordinates.

60. The result of the last Article may also be obtained thus. Suppose that the given forces have a single resultant acting at the point (x', y'), and giving components X' and Y' parallel to the co-ordinate axes. It follows that the given forces will, with $-X'$, $-Y'$ acting at the point (x', y'), form a system in equilibrium. Hence, by Art. 57,

$$\Sigma X - X' = 0, \quad \Sigma Y - Y' = 0, \quad G - Y'x' + X'y' = 0.$$

Of these three equations the first determines X', the second Y', and the third assigns a relation between x' and y', which is in fact the equation to the line in which the single resultant acts and at any point of which it may be supposed to act. If ΣX and ΣY both vanish, it is impossible to find values of x' and y' that satisfy the last equation of the three, so long as G does not vanish; this shews that if the forces reduce to a couple, it is impossible to find a single force equivalent to them.

61. In Art. 56, we have for the moment of the force P_1 about the origin the expression

$$P_1\,(x_1 \sin\alpha_1 - y_1 \cos\alpha_1),$$

and this we may express by

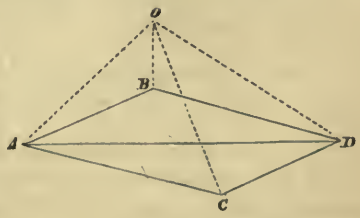
$$Y_1 x_1 - X_1 y_1.$$

Since X_1 and Y_1 are the rectangular components of P_1, we see by comparing the two expressions that the moment of a force about any origin is equal to the algebraical sum of the moments of its rectangular components about the same origin. (See Art. 55.) There are many such theorems connected with moments, and the demonstration of some of them

is facilitated by observing that according to the definition of a moment, it may be geometrically represented by twice the area of the triangle having for its base the straight line which represents the force and for its vertex the point about which moments are taken. For example, we may prove the theorem which we have already deduced.

62. *The algebraical sum of the moments of two component forces with respect to any point in the plane containing the two forces is equal to the moment of the resultant of the two forces.*

Let AB, AC represent two component forces; complete the parallelogram and draw the diagonal AD representing the resultant force.



(1) Let O, the point about which the moments are to be taken, fall *without* the angle BAC and that which is vertically opposite to it. Join OA, OB, OC, OD.

The triangle OAC having for its base AC and for its height the perpendicular from O on AC is equivalent to a triangle having AC for its base and for its height the perpendicular from B on AC, together with a triangle having BD for its base and for its height the perpendicular from O on BD. This is obvious since BD is equal and parallel to AC, and the perpendicular from O on AC is equal to the perpendicular from O on BD together with the perpendicular from B on AC. Hence we have

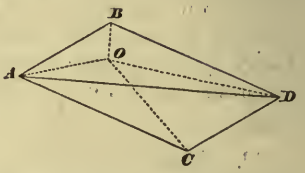
$$\triangle AOC = \triangle BOD + \triangle ACD.$$

Hence, adding the triangle AOB, we have

$$\triangle AOC + \triangle AOB = \triangle BOD + \triangle ABD + \triangle AOB = \triangle AOD;$$

that is, the moment of AC + the moment of AB = the moment of AD.

(2) Let O fall *within* the angle BAC or its vertically opposite angle.



$$\triangle AOC = \triangle ABD - \triangle BOD$$
$$= \triangle AOB + \triangle AOD.$$

Therefore

$$\triangle AOD = \triangle AOC - \triangle AOB;$$

that is, the moment of AD = the moment of AC − the moment of AB. As the moments of AC and AB about O are now of *opposite* characters, the moment of the resultant is still equal to the algebraical sum of the moments of the components.

The proposition may also be readily shewn in the case where the two component forces are *parallel;* see Art. 37.

Another proof of the proposition may be derived from the fact that the moment of a force represented by a line AP about a point O is $AP \cdot AO \sin OAP$. Expressing the three moments in this form and dividing out by AO the proposition is reduced to the assertion that the projection of AD on a line perpendicular to AO is the sum of the projections of AC and CD.

The corresponding theorem for any number of forces acting at a point in one plane can be proved analytically as follows. With the notation of Art. 56, we have

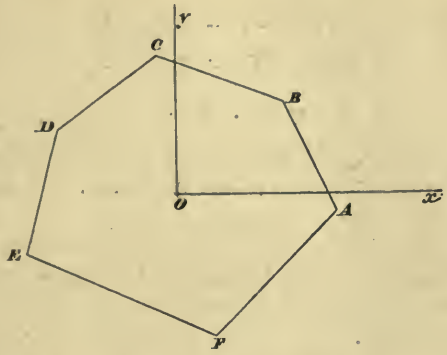
$$R \cos a = \Sigma P_1 \cos \alpha_1, \qquad R \sin a = \Sigma P_1 \sin \alpha_1.$$

Multiply the first equation by $l \sin \phi$, the second by $l \cos \phi$, and subtract. We thus obtain

$$Rl \sin (\phi - a) = \Sigma P_1 l \sin (\phi - \alpha_1),$$

which, if we make l stand for the distance of the point about which moments are taken from the common point of application of the forces, and ϕ for the inclination of l to the axis of x, expresses that the moment of the resultant is the sum of the moments of the components.

63. *Forces are represented in magnitude and position by the sides of a plane polygon taken one way round; required the resultant.*



Let the sides of the figure $ABCDEF$ represent the forces in magnitude and position; the first force being supposed to act in the straight line AB from A towards B, the second in the straight line BC from B towards C, and so on.

As in Art. 56, the forces may be replaced by a resultant force at an arbitrary origin O and a couple. The former is composed of all the forces AB, BC,..... moved each parallel to itself up to O; the resultant force consequently vanishes by Art. 21.

The moment of the resultant couple is the sum of the moments of the component couples, and is therefore represented by twice the triangle AOB + twice the triangle BOC + ...; that is, by *twice the area of the polygon.* Hence the forces reduce to a resultant couple measured by twice the area of the polygon.

We may observe that the algebraical sum of the moments of the two forces which form a couple is the same about whatever point it be taken; it is in fact equal to the *moment of the couple.*

64. If the sum of the moments of the forces P_1, P_2, P_3,... be required about a point whose co-ordinates are h, k instead of about the origin, we must in the expression for G, in

Art. 56, put $x_1 - h$, $x_2 - h$, ... for x_1, x_2, ... respectively, an $y_1 - k$, $y_2 - k$, ... for y_1, y_2, ... respectively. Hence, denotin the result by G_1, we have

$$\begin{aligned} G_1 &= \Sigma \{Y(x-h) - X(y-k)\} \\ &= k\Sigma X - h\Sigma Y + \Sigma(Yx - Xy) \\ &= k\Sigma X - h\Sigma Y + G. \end{aligned}$$

Hence the value of G_1 depends in general on the situatio of the point about which we take moments. If, however,

$$k\Sigma X - h\Sigma Y = \text{a constant},$$

that is, if the point (h, k) move along any straight lin parallel to the direction of the resultant force R, then G remains unchanged.

If three different points exist with respect to which th sum of the moments vanishes, we have three equations

$$\begin{aligned} k_1\Sigma X - h_1\Sigma Y + G &= 0, \\ k_2\Sigma X - h_2\Sigma Y + G &= 0, \\ k_3\Sigma X - h_3\Sigma Y + G &= 0. \end{aligned}$$

Hence we deduce

$$\begin{aligned} (k_1 - k_2)\Sigma X &= (h_1 - h_2)\Sigma Y, \\ (k_2 - k_3)\Sigma X &= (h_2 - h_3)\Sigma Y. \end{aligned}$$

Unless the point (h_1, k_1), the point (h_2, k_2), and the poin (h_3, k_3) lie in a straight line, it is impossible that

$$\frac{k_1 - k_2}{h_1 - h_2} = \frac{k_2 - k_3}{h_2 - h_3};$$

we must therefore have

$$\Sigma X = 0, \quad \Sigma Y = 0, \quad G = 0.$$

Hence if the sum of the moments of a system of forces i one plane vanish with respect to three points in the plane n in a straight line, that system is in equilibrium.

When a system of forces in one plane can be reduced to a single force, we have found in Art. 59 that the equation to its line of action is

$$x'\Sigma Y - y'\Sigma X = \Sigma (Yx - Xy).$$

This may be written

$$\Sigma \{Y(x' - x) - X(y' - y)\} = 0.$$

The equation to the line of action of the resultant thus in fact determines the locus of the points for which the *algebraical sum* of the moments of the forces is zero.

65. Hitherto we have supposed our axes rectangular. If they are oblique and inclined at an angle ω, we may shew, as in Art. 56, that a system of forces in one plane may be reduced to ΣX along the axis of x, ΣY along the axis of y, and a couple the moment of which is $\sin\omega\Sigma (Yx - Xy)$. The latter part will be easily obtained, since the moment of the force P_1 is equivalent to the algebraical sum of the moments of its components X_1 and Y_1; and the perpendicular on the former from the origin is $y_1 \sin\omega$, and on the latter $x_1 \sin\omega$.

The conditions for equilibrium are, as before,

$$\Sigma X = 0, \quad \Sigma Y = 0, \quad \Sigma (Yx - Xy) = 0.$$

The following Examples may be solved by means of the principles given in the preceding Articles. When different rigid bodies occur in a question, the equations of Art. 57 must hold with respect to *each*, in order that there may be equilibrium. In cases where only *three* forces act on a body, it is often convenient to use the proposition of Art. 58. Since by Art. 57 the moments of the forces with respect to *any* origin must vanish, we may, if we please, take different origins and form the corresponding equation for each. See Art. 64.

In some of the Examples we anticipate the results of the subsequent Chapters so far as to assume that the weight of any body acts through a definite and known point, which is the centre of gravity of the body. When two bodies are in contact it is assumed that whatever force one exerts on the other the latter exerts an equal and opposite force on the former; if the bodies are smooth this force acts in the direction

of the common normal to the surfaces at the point of contact. We restrict ourselves to the supposition of smooth bodies until Chapter X.

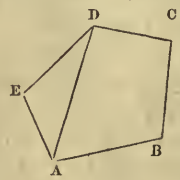
In attempting to solve the problems the student will find it advisable when the system involves more than one body to confine his attention to one at a time of those bodies which are capable of motion, and to be careful to take into consideration *all* the forces which act on that body. When bodies are in contact some letter should be used to denote the mutual force between them, and the magnitude of this force must be found from the equations of equilibrium of the body or bodies which are capable of motion. And when two of the bodies are connected by a string a letter should be used to denote the tension of the string, and the magnitude of the tension must be found from the conditions of equilibrium of the body or bodies which are capable of motion. Beginners often fall into error by *assuming* incorrect values for the tensions of strings and the mutual forces between bodies in contact, instead of determining the correct values from the equations of equilibrium.

We will give here two propositions, respecting forces acting in a plane, which involve important results.

I. Forces act at the middle points of the sides of a rigid polygon in the plane of the polygon; the forces act at right angles to the sides, and are respectively proportional to the sides in magnitude: shew that the forces will be in equilibrium if they all act inwards or all act outwards.

The result here enunciated has been already shewn to be true in the case of a triangle; see the Proposition III. at the end of Chapter II.; the general proposition is obtained by an inductive method.

Suppose for example that the proposition were known to be true for a four-sided figure also; then we can shew that it must be true for a five-sided figure. Let *ABCDE* be a five-sided figure; and let forces act at the middle points of the sides in the plane of the figure, at right angles to the sides and respectively proportional to the sides



in magnitude: suppose for the sake of distinctness that the forces all act outwards.

Join AD. By hypothesis a certain system of forces acting outwards on the four-sided figure $ABCD$ would be in equilibrium; and from this it follows that the assigned forces acting on AB, BC, CD must be equivalent to a single force acting at the middle point of AD, towards the inside of the four-sided figure $ABCD$, proportional to AD in magnitude.

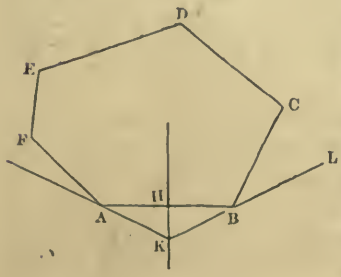
Also the assigned forces acting on DE, EA must in like manner be equivalent to a single force acting at the middle point of AD, towards the inside of the triangle AED, proportional to AD in magnitude.

Hence the two single forces balance each other; and the system is in equilibrium.

In this manner, knowing that the result is true for a triangle, we can shew in succession that it is true for a figure of four, five, six,... sides.

II. Rigid rods without weight are joined together by smooth hinges at their extremities, so as to form a plane polygon. Forces act at the middle points of the sides of the polygon in the plane of the polygon; the forces act at right angles to the sides, and are respectively proportional to the sides in magnitude; shew that, if the forces all act inwards or all act outwards, when there is equilibrium, a circle can be described round the polygon.

Let $ABCDEF$ represent the polygon. Consider one of the rods as AB. This rod is acted on by a force at the middle point H at right angles to AB, and by actions from the hinges at A and B. The former force is proportional to AB, and may be denoted by $\mu . AB$. The three forces must meet at a point, suppose K; then by resolving parallel to AB, we find that the actions at A and B must be equal; we will denote



them by R. Resolve the forces at right angles to AB: thus

$$\mu AB = 2R \sin ABK.$$

The action at B on the rod BC is equal and opposite to that on the rod BA; hence we obtain in the same manner

$$\mu BC = 2R \sin CBL.$$

Therefore
$$\frac{\sin ABK}{\sin CBL} = \frac{AB}{BC} = \frac{\sin ACB}{\sin CAB}.$$

This shews that KBL touches at B the circle described round ABC.

Similarly AK touches at A the circle described round BAF.

But AK and BK are equally inclined to AB. Hence the two circles must coincide; that is, the points F, A, B, C lie on the circumference of a circle.

In this way we shew that any four consecutive angular points of the polygon lie on the circumference of a circle; and hence it follows that all the angular points must lie on the circumference of the same circle.

It will be seen from the preceding results that the action at every hinge is the same, and is denoted by the product of μ into the radius of the circle described round the polygon.

EXAMPLES.

1. $ABCD$ is a quadrilateral and is acted on by forces which are represented in magnitude and direction by AB, AD, CB, CD; shew that the resultant coincides in direction with the straight line which joins the middle points of the diagonals AC, BD, and is represented in magnitude by four times this straight line.

2. Forces proportional to the sides of an isosceles triangle act along the sides of the triangle, those acting along the equal sides tending from the vertex; find the magnitude and position of their resultant.

Result. The required resultant is represented by a straight line which passes through the middle point of the base of the

triangle, is parallel to one of the sides, and double that side in length.

3. The upper end of a uniform heavy rod rests against a smooth vertical wall; one end of a string is fastened to the lower end of the rod and the other end of the string is fastened to the wall; the position of the rod being given, find the point of the wall to which the string must be fastened, in order that the rod may be in equilibrium.

4. A uniform heavy rod is placed across a smooth horizontal rail, and rests with one end against a smooth vertical wall, the distance of which from the rail is $\frac{1}{16}$th of the length of the rod; find the position of equilibrium.

Result. The rod makes an angle of 60° with the horizon.

5. ABC is a triangular lamina; AD, BE, CF are the perpendiculars on the sides, and forces represented by the straight lines BD, CD, CE, AE, AF, BF are applied to the lamina; shew that their resultant will pass through the centre of the circle described about the triangle.

6. AB, AC are two equal beams connected by a hinge at A, and by a string joining the extremities B and C: AB is fixed vertically, and a sphere of given weight and radius is supported between the two beams: find the pressure of the sphere on each beam, and the tension of the string.

7. An elliptic lamina is acted on at the extremities of pairs of conjugate diameters by forces in its own plane tending outwards, and normal to its edge: shew that there will be equilibrium if the force at the end of every diameter be proportional to the conjugate diameter.

8. A heavy sphere hangs from a peg by a string whose length is equal to the radius, and it rests against another peg vertically below the former, the distance between the two being equal to the diameter. Find the tension of the string and the pressure on the lower peg.

Results. The tension is equal to the weight of the sphere and the pressure to half the weight of the sphere.

9. Two equal rods without weight are connected at their middle points by a pin which allows free motion in a vertical plane; they stand upon a horizontal plane, and their upper extremities are connected by a thread which carries a weight. Shew that the weight will rest half way between the pin and the horizontal line joining the upper ends of the rods.

10. Two equal circular discs with smooth edges, placed on their flat sides in the corner between two smooth vertical planes inclined at a given angle, touch each other in the straight line bisecting the angle. Find the radius of the least disc which may be pressed between them without causing them to separate.

11. A flat semicircular board with its plane vertical and curved edge upwards rests on a smooth horizontal plane, and is pressed at two given points of its circumference by two beams which slide in smooth vertical tubes; find the ratio of the weights of the beams that the board may be in equilibrium.

12. Two smooth cylinders with their axes horizontal, of equal radii, just fit in between two parallel vertical walls, and rest on a smooth horizontal plane without pressing against the walls; if a third cylinder be placed on the top of them, find the resulting pressure against either wall.

13. A smooth circular ring rests on two pegs not in the same horizontal plane; find the pressure on each peg.

14. Two spheres are supported by strings attached to a given point, and rest against one another; find the tensions of the strings.

15. Two unequal smooth spheres, connected by a string, are laid upon the surface of a cylinder, the string being so short as not to touch the cylinder; determine the position of rest and the tension of the string.

16. A heavy regular polygon is attached to a smooth vertical wall by a string which is fastened to the middle point of one of its sides; the plane of the polygon is vertical and perpendicular to the wall, and one of the extremities of the side to which the string is attached rests against the wall; shew that whatever be the length of the string when the

polygon is in equilibrium, the tension of the string and the pressure on the wall are constant.

17. A straight rod without weight is placed between two pegs, and forces P and Q act at its extremities in parallel directions inclined to the rod; required the conditions under which the rod will be at rest and the pressures on the pegs.

Result. P and Q must be equal and in opposite directions.

18. Forces P, Q, R, S act along the sides of a rectangle; find the direction of the resultant force.

19. Two equal weights P, P are attached to the ends of two strings which pass over the same smooth peg and have their other extremities attached to the ends of a beam AB, the weight of which is W; shew that when there is equilibrium the inclination of the beam to the horizon $= \tan^{-1}\left(\frac{a-b}{a+b}\tan\alpha\right)$; a, b being the distances of the centre of gravity of the beam from its ends, and $\sin\alpha = \frac{W}{2P}$.

20. A square is placed with its plane vertical between two small pegs which are in the same horizontal line; shew that it will be in equilibrium when the inclination of one of its edges to the horizon $= \frac{1}{2}\sin^{-1}\frac{a^2-c^2}{c^2}$, $2a$ being the length of a side of the square, and c the distance between the pegs. Shew that the equilibrium will not be affected by the application of any force which bisects the straight line joining the pegs and passes through the lowest point of the square.

21. One end of a string is fixed to the extremity of a smooth uniform rod, and the other to a ring without weight which passes over the rod, and the string is hung over a smooth peg. Determine the least length of the string for which equilibrium is possible, and shew that the inclination of the rod to the vertical cannot be less than 45°.

22. A string nine feet long has one end attached to the extremity of a smooth uniform heavy rod two feet in length,

and at the other end carries a ring without weight which slides on the rod. The rod is suspended by means of the string from a smooth peg; shew that if θ be the angle which the rod makes with the horizon, then $\tan\theta = 3^{-\frac{1}{2}} - 3^{-\frac{3}{2}}$.

23. A square rests with its plane perpendicular to a smooth wall, one corner being attached to a point in the wall by a string whose length is equal to a side of the square; shew that the distances of three of its angular points from the wall are as 1, 3, and 4.

24. One end of a beam, whose weight is W, is placed on a smooth horizontal plane; the other end, to which a string is fastened, rests against another smooth plane inclined at an angle α to the horizon; the string passing over a pully at the top of the inclined plane hangs vertically, supporting a weight P. Shew that the beam will rest in all positions if a certain relation hold between P, W, and α.

25. If a weight be suspended from one extremity of a rod moveable about the other extremity A, which remains fixed, and a string of given length be attached to any point B in the rod, and also to a fixed point C above A, and in the same vertical line with it, then the tension of the string varies inversely as the distance AB.

26. One end of a uniform beam is placed on the ground against a fixed obstacle, and to the other end is attached a string which runs in a horizontal direction to a fixed point in the same vertical line as the obstacle, and, passing freely over it, is kept in tension by a weight W suspended at its extremity, the beam being thus held at rest at an inclination of 45° to the horizon. Shew that if the string were attached to the centre instead of to the end of the beam, and passed over the same fixed point, a weight $= \sqrt{2}W$ would keep the beam in the same position.

27. Two equal beams AB, AC connected by a hinge at A are placed in a vertical plane with their extremities B, C resting on a horizontal plane; they are kept from falling by strings connecting B and C with the middle points of the opposite beams; shew that the ratio of the tension of each

string to the weight of each beam

$$= \tfrac{1}{8}\sqrt{(9\cot^2\theta + 1)},$$

θ being the inclination of each beam to the horizon.

28. One end of a string is attached to a beam at the point B, and the other end is fastened to the highest point A of a fixed sphere of radius r. If the points of contact of the beam and string trisect the quadrant AC, shew that the distance between B and the centre of gravity of the beam must be $2r(2-\sqrt{3})$.

29. A heavy rod can turn freely about a fixed hinge at one extremity, and it carries a heavy ring which is attached to a fixed point in the same horizontal plane with the hinge by means of a string of length equal to the distance between the point and the hinge. Find the position in which the rod will rest.

30. Two equal heavy beams of sufficient length, and connected by a hinge, are supported by two smooth pegs in the same horizontal line; a sphere is placed between them, determine the position of equilibrium.

31. Forces P, Q, R act along the sides BC, CA, AB of a triangle, and their resultant passes through the centres of the inscribed and circumscribed circles; shew that

$$P : Q : R :: \cos B - \cos C \ : \ \cos C - \cos A \ : \ \cos A - \cos B.$$

32. Find the position of equilibrium of a uniform beam resting in a vertical plane with one end pressing against a vertical wall, and the other end supported by the convex arc of a parabola in the vertical plane, whose vertex is at the foot of the wall and axis horizontal.

33. A uniform beam PQ of given weight and length rests in contact with a fixed vertical circle whose vertical diameter is AB, in such a manner that strings AP, BQ attached to the rod and circle are tangents to the circle at the points A and B. Find the tensions of the strings, and shew that the conditions of the problem require that the inclination of the beam to the vertical must be less than $\sin^{-1}\dfrac{\sqrt{5}-1}{2}$.

34. Shew that no uniform rod can rest partly within and partly without a fixed smooth hemispherical bowl at an inclination to the horizon greater than $\sin^{-1}\frac{1}{\sqrt{3}}$.

35. The sides of a rigid plane polygon are acted on by forces at right angles to the sides and proportional to them in magnitude, all the forces acting in the plane of the polygon, and being inwards; also the sides taken in the same order are severally divided by the points of application in the constant ratio of p to q; shew that the system of forces is equivalent to a couple whose moment is

$$\frac{\mu(p-q)}{2(p+q)}\Sigma a^2,$$

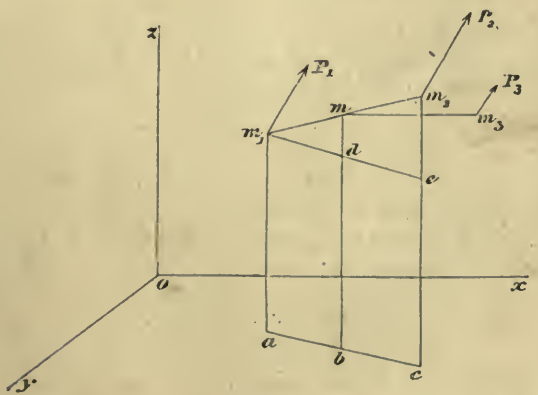
where μa represents the force applied to any side a of the polygon, and Σa^2 the sum of the squares of the sides.

CHAPTER V.

FORCES IN DIFFERENT PLANES.

66. *To find the magnitude and direction of the resultant of any number of parallel forces acting on a rigid body, and to determine the centre of parallel forces.*

Let the points of application of the forces be referred to a system of rectangular co-ordinate axes. Let m_1, m_2, ... be the



points of application; let x_1, y_1, z_1 be the co-ordinates of the first point, x_2, y_2, z_2 those of the second, and so on; let P_1, P_2, ... be the forces acting at these points, those being reckoned positive which act in the direction of P_1, and those negative which act in the opposite direction.

Join m_1m_2; and take the point m on m_1m_2 such that

$$m_1m = \frac{P_2}{P_1 + P_2} \cdot m_1m_2;$$

then the resultant of P_1 and P_2 is P_1+P_2, and it acts through m parallel to P_1. (Art. 37.)

Draw m_1a, mb, m_2c perpendicular to the plane of (x, y), meeting that plane at a, b, c; draw m_1de parallel to abc meeting mb at d and m_2c at e. Then, by similar triangles,

$$\frac{m_1m}{m_1m_2}=\frac{md}{m_2e}=\frac{mb-z_1}{z_2-z_1},$$

therefore
$$mb-z_1=\frac{P_2}{P_1+P_2}(z_2-z_1);$$

therefore
$$mb=\frac{P_1z_1+P_2z_2}{P_1+P_2}.$$

This gives the ordinate parallel to the axis of z of the point of application of the resultant of P_1 and P_2.

Then supposing P_1 and P_2 to be replaced by P_1+P_2 acting at m, the resultant of P_1+P_2 and P_3 is $P_1+P_2+P_3$, and the ordinate of its point of application

$$=\frac{(P_1+P_2)\,mb+P_3z_3}{P_1+P_2+P_3}=\frac{P_1z_1+P_2z_2+P_3z_3}{P_1+P_2+P_3};$$

and this process may be extended to any number of parallel forces. Let R denote the resultant force and $\bar{z}$ the ordinate of its point of application; then

$$R=\Sigma P,\quad \bar{z}=\frac{\Sigma Pz}{\Sigma P}.$$

Similarly, if $\bar{x}$, $\bar{y}$ be the other co-ordinates of the point of application of the resultant,

$$\bar{x}=\frac{\Sigma Px}{\Sigma P};\quad \bar{y}=\frac{\Sigma Py}{\Sigma P}.$$

The values of $\bar{x}$, $\bar{y}$, $\bar{z}$ are independent of the angles which the directions of the forces make with the axes. Hence if these directions be turned about the points of application of the forces, their parallelism being preserved, the point of application of the resultant will not move. For this reason this point is called the *centre of the parallel forces.*

67. The moment of a force with respect to a plane is the product of the force into the perpendicular distance of its point of application from the plane.

In consequence of this definition, the equations for determining the position of the centre of parallel forces shew that *the sum of the moments of any number of parallel forces with respect to any plane is equal to the moment of their resultant.*

It is to be noted that whereas the moment of a force with respect to a point or with respect to a line (Art. 74), measures the tendency of the force to produce rotation round that point or line, the moment of a force with respect to a plane does not measure any tendency to produce rotation, unless the force is parallel to the plane, in which case it measures the tendency to produce rotation round any line (in the plane) perpendicular to the force.

68. If the parallel forces all act in the *same* direction the expression ΣP cannot vanish; hence the values of the co-ordinates of the *centre of parallel forces* found in Art. 66 cannot become infinite or indeterminate, and we are certain that the centre exists. But if some of the forces are positive and some negative, ΣP *may* vanish, and the results of Art. 66 become nugatory. In this case, since the sum of the *positive* forces is equal to the sum of the *negative* forces, the resultant of the former will be equal to the resultant of the latter. Hence the resultant of the whole system of forces is a *couple*, unless the resultant of the positive forces should happen to lie in the same straight line as the resultant of the negative forces.

We shall give another method of reducing a system of parallel forces.

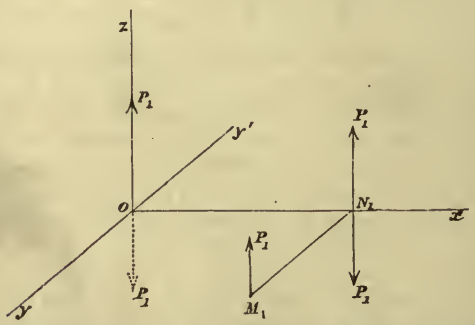
69. *To find the resultant of a system of parallel forces acting upon a rigid body.*

Let P_1, P_2, ... denote the forces. Take the axis of z parallel to the forces. Let the plane of (x, y) meet the line of action of P_1 at M_1, and suppose x_1, y_1 the co-ordinates of this point.

Draw M_1N_1 perpendicular to the axis of x meeting it at N_1. At the origin O, and also at N_1, apply two forces each equal

and parallel to P_1 and in opposite directions. Hence the force P_1 at M_1 is equivalent to the following system,

(1) P_1 at O;
(2) a couple formed of P_1 at M_1 and P_1 at N_1;
(3) a couple formed of P_1 at N_1 and P_1 at O.



The moment of the first couple is P_1y_1, and this couple, without altering its effect, may be transferred to the plane of (y, z), which is parallel to its original plane. The moment of the second couple is P_1x_1, and the couple is in the plane of (x, z). Thus far we have neglected their signs, which we now proceed to determine, according to the rules laid down in Art. 40.

Assuming x_1 and y_1 to be positive, and P_1 to act in the direction of the positive axis of z, the first couple tends to turn the body from Oy towards Oz. This is the positive direction; hence the moment of this couple is to be taken as $+P_1y_1$ and its axis is to be drawn in the direction Ox. The second couple tends to turn the body from Ox towards Oz. This is the negative direction. The moment of this couple is therefore to be taken as $-P_1x_1$ and its axis is to be drawn in the direction Oy'.

If we effect a similar transformation of all the forces, we have, as the resultant of the system, the following system,

(1) a force ΣP acting at O;
(2) a couple ΣPy in the plane of (y, z);
(3) a couple $-\Sigma Px$ in the plane of (x, z).

The two couples (2), (3) will compound into a single couple. Let G denote its moment regarded as essentially positive, and α the inclination of its axis to Ox. Then we have

$$G^2 = (\Sigma Px)^2 + (\Sigma Py)^2;$$

$$\cos\alpha = \frac{\Sigma Py}{G}; \quad \sin\alpha = \frac{-\Sigma Px}{G}.$$

This couple G and the force ΣP can, by Art. 59, be reduced to a single force ΣP at a distance $\frac{G}{\Sigma P}$ from the axis of z.

Definition. By *a couple round a line* we mean a couple in a plane perpendicular to the line. Thus a couple *in the plane of* (y, z) is a couple *round the axis of* x.

70. *To find the conditions of equilibrium of a system of parallel forces acting on a rigid body.*

A system of parallel forces can always be reduced to a single force applied at an arbitrary point, and a couple. Since these are in the same plane, it can be shewn as in Art. 54 that for equilibrium we must have

$$\Sigma P = 0, \text{ and } G = 0;$$

the latter requires that

$$\Sigma Px = 0, \text{ and } \Sigma Py = 0.$$

Hence *a system of parallel forces acting on a rigid body will be in equilibrium if the sum of the forces vanishes, and also the sum of the moments vanishes with respect to two planes at right angles to each other and parallel to the forces.*

Conversely, if the forces are in equilibrium the sum of the forces will vanish, and also the sum of the moments with respect to any two planes at right angles to each other and parallel to the forces.

71. When $\Sigma P = 0$, the forces reduce to a couple of which the moment is G. When ΣP is not $= 0$, the forces can always be reduced to a single force; this has already appeared in Arts. 66 and 69, and may also be shewn thus. The forces

will reduce to a resultant R acting at the point (x', y'), parallel to the original forces, provided a force $-R$ acting at this point will with the given forces maintain equilibrium. The necessary and sufficient conditions for this are, by Art. 70,

$$\Sigma P - R = 0, \quad \Sigma Px - Rx' = 0, \quad \Sigma Py - Ry' = 0.$$

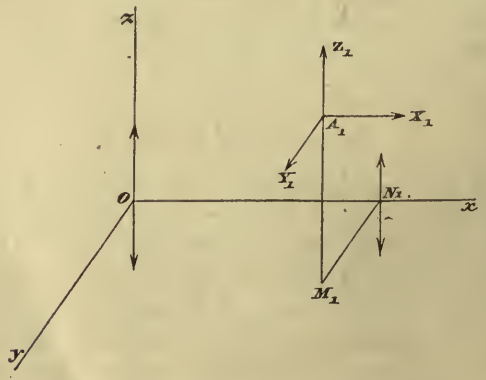
Hence $$R = \Sigma P, \quad x' = \frac{\Sigma Px}{\Sigma P}, \quad y' = \frac{\Sigma Py}{\Sigma P}.$$

These results agree with those of Arts. 66 and 69.

72. *To find the resultants of any number of forces acting on a rigid body in any directions.*

Let the forces be referred to three rectangular axes Ox, Oy, Oz; and suppose $P_1, P_2, P_3, \ldots$ the forces; let x_1, y_1, z_1 be the co-ordinates of the point of application of P_1; let x_2, y_2, z_2 be the co-ordinates of the point of application of P_2; and so on.

Let A_1 be the point of application of P_1; resolve P_1 into components X_1, Y_1, Z_1, parallel to the co-ordinate axes. Let



the line of action of Z_1 meet the plane of (x, y) at M_1, and draw M_1N_1 perpendicular to Ox. Apply at N_1 and also at O two forces each equal and parallel to Z_1, and in opposite directions. Hence Z_1 at A_1 or M_1 is equivalent to Z_1 at O, and two couples, one having its moment $= Z_1 . N_1M_1$, and

lying in a plane perpendicular to the axis of x, and the other having its moment $= Z_1 . ON_1$ and lying in a plane perpendicular to the axis of y.

We are to regard those couples as positive which tend to turn the body round the axis of x from y to z, also those which tend to turn the body round the axis of y from z to x, and those which tend to turn the body round the axis of z from x to y.

Hence Z_1 is replaced by Z_1 at O, a couple $Z_1 y_1$ round Ox, and a couple $-Z_1 x_1$ round Oy. Similarly X_1 may be replaced by X_1 at O, a couple $X_1 z_1$ round Oy, and a couple $-X_1 y_1$ round Oz. And Y_1 may be replaced by Y_1 at O, a couple $Y_1 x_1$ round Oz, and a couple $-Y_1 z_1$ round Ox. Therefore the force P_1 may be replaced by X_1, Y_1, Z_1 acting at O, and the couples of which the moments are, by Art. 47,

$$Z_1 y_1 - Y_1 z_1 \text{ round } Ox,$$
$$X_1 z_1 - Z_1 x_1 \text{ ..... } Oy,$$
$$Y_1 x_1 - X_1 y_1 \text{ ..... } Oz.$$

By a similar resolution of all the forces we shall have them replaced by the forces

$$\Sigma X, \quad \Sigma Y, \quad \Sigma Z,$$

acting at O along the axes, and the couples

$$\Sigma (Zy - Yz) = L \text{ suppose, round } Ox,$$
$$\Sigma (Xz - Zx) = M \text{ ....., ... } Oy,$$
$$\Sigma (Yx - Xy) = N \text{ ....., ... } Oz.$$

Let R be the resultant of the forces which act at O; a, b, c the angles its direction makes with the axes; then, by Art. 24,

$$R^2 = (\Sigma X)^2 + (\Sigma Y)^2 + (\Sigma Z)^2,$$
$$\cos a = \frac{\Sigma X}{R}, \quad \cos b = \frac{\Sigma Y}{R}, \quad \cos c = \frac{\Sigma Z}{R}.$$

Let G be the moment of the couple which is the resultant of the three couples L, M, N; λ, μ, ν the angles its axis makes with the co-ordinate axes; then, by Art. 49,

$$G^2 = L^2 + M^2 + N^2,$$

$$\cos\lambda = \frac{L}{G}, \quad \cos\mu = \frac{M}{G}, \quad \cos\nu = \frac{N}{G}.$$

For the convention as to the signs of L, M, N see Art. 40.

73. *To find the conditions of equilibrium of any number of forces acting on a rigid body in any directions.*

A system of forces acting on a rigid body can always be reduced to a single force and a couple. Since these cannot balance each other* and cannot separately maintain equilibrium they must both vanish. Hence $R = 0$, and $G = 0$;

therefore $$(\Sigma X)^2 + (\Sigma Y)^2 + (\Sigma Z)^2 = 0,$$

and $$L^2 + M^2 + N^2 = 0.$$

These lead to the six conditions

$$\Sigma X = 0, \quad \Sigma Y = 0, \quad \Sigma Z = 0,$$

$$\Sigma (Zy - Yz) = 0, \quad \Sigma (Xz - Zx) = 0, \quad \Sigma (Yx - Xy) = 0.$$

74. A verbal enunciation may be given of the last three equations by means of a new definition. For the sake of convenience, we repeat two definitions already given in Arts. 55 and 67.

Moment of a force with respect to a *point.* The moment of a force with respect to a point is the product of the force into the perpendicular from the point on the line of action of the force.

Moment of a force with respect to a *plane.* The moment of a force with respect to a plane is the product of the force into the distance of its point of application from the plane.

Moment of a force with respect to a *straight line.* Resolve the force into two components respectively parallel and perpendicular to the straight line; the product of the component perpendicular to the line into the shortest distance between

* If they balanced each other, the couple would be equivalent to a force equal and opposite to the given force. But it is absurd to suppose that a couple can be equivalent to a force, since equal and parallel couples are equivalent, and equal and parallel forces are not equivalent.

the straight line and the line of action of this component is called the *moment* of the force with respect to the straight line.

Hence the moment of a force with respect to a straight line is equal to the moment of the component of the force perpendicular to the straight line with respect to the *point* at which a plane drawn through this component perpendicular to the straight line meets the straight line. Hence, by Art. 62, the moment of the force may be found by taking the sum of the moments of any two forces into which the perpendicular component may be resolved.

If the force is *parallel* to the given straight line, its moment about the straight line is zero. If the force is *perpendicular* to the given straight line, its moment about the straight line is the product of the force into the shortest distance between it and the given straight line.

75. Suppose we require the moment of the force P_1 about the axis of z; we resolve P_1 into the forces Z_1 parallel to the axis of z and Q_1 perpendicular to the axis of z, where Q_1 is itself the resultant of X_1 and Y_1. The moment of Q_1 with respect to the axis of z is equal to the algebraical sum of the moments of its components X_1 and Y_1; that is, to $Y_1x_1 - X_1y_1$. Hence N in Art. 72 denotes the *sum of the moments of the forces round the axis of z*, and similar meanings arise for L and M.

Hence, the forces acting on a rigid body will be in equilibrium if *the sums of the resolved parts of the forces parallel to three straight lines at right angles to each other vanish, and the sums of the moments of the forces with respect to these straight lines also vanish.*

Conversely, if the forces are in equilibrium, the sum of the resolved parts of the forces in any direction will vanish, and also the sum of the moments of the forces with respect to any straight line.

76. Without employing the rectangular axes Ox, Oy, Oz, we might have effected the reduction of the given forces by the more direct process of introducing at O two forces each equal and parallel to P_1, and in opposite directions, thus resolving P_1 into a force at O and a couple. Proceeding thus

with each of the given forces, we obtain a number of forces acting at O and a number of couples. The forces at O will compound into a single force R, and the couples into a single couple G. This is the simplest way of proving that any set of forces applied to a rigid body are in general equivalent to a single force applied at an arbitrary point, together with a couple. For calculating the magnitudes of this force and couple, the method of rectangular axes above employed is usually the most convenient.

Both R and G are perfectly determinate when the origin O is given. For if R_1 and G_1 are together equivalent to R_2 and G_2, the two latter reversed will equilibrate the two former. R_1 and the reversed R_2 will give a force R_3 at O, and G_1 with the reversed G_2 will give a couple G_3, and for equilibrium both R_3 and G_3 must vanish as shewn in Art. 73. That is, R_1 must be identical with R_2, and G_1 with G_2.

The value of G will depend upon the position of the origin O; but R, being the resultant of all the given forces supposing them applied at a point, is independent of the choice of origin.

Since the direction of the co-ordinate axes is arbitrary, suppose the axis of x to coincide with the axis of G; then $M=0$, $N=0$, and L and G are identical.

Hence *G is equal to the sum of the moments of the given forces with respect to the straight line which is the axis of G.*

77. Suppose a force P acting at the point (x, y, z), and let X, Y, Z be its components parallel to the axes. Then, by Art. 72, P at the point (x, y, z) is equivalent to P at the origin, together with the couples $Zy-Yz$, $Xz-Zx$, $Yx-Xy$ round the axes of x, y, z respectively. Let H be the resultant couple, r the distance of the point (x, y, z) from the origin, and α the angle between r and P; then

$$\begin{aligned} H^2 &= (Zy-Yz)^2+(Xz-Zx)^2+(Yx-Xy)^2 \\ &= (x^2+y^2+z^2)(X^2+Y^2+Z^2)-(xX+yY+zZ)^2 \\ &= r^2P^2-r^2P^2\left(\frac{x}{r}\frac{X}{P}+\frac{y}{r}\frac{Y}{P}+\frac{z}{r}\frac{Z}{P}\right)^2 \\ &= r^2P^2(1-\cos^2\alpha), \end{aligned}$$

therefore $H = rP\sin\alpha$.

Thus, as we might have anticipated, H is the moment of the couple formed by P at the point (x, y, z) and a force at the origin equal to P and acting in a parallel and opposite direction. Hence G is the couple formed by compounding the couples similar to H arising from all the forces of the system.

78. As an example of Art. 73 we may take the case in which all the forces are *parallel*. Let α, β, γ be the angles which the direction of the forces $P_1, P_2, \ldots\ldots$ makes with the axes. Then the equations of equilibrium reduce to

$$\Sigma P = 0,$$

$$\Sigma P\,(y\cos\gamma - z\cos\beta) = 0,$$

$$\Sigma P\,(z\cos\alpha - x\cos\gamma) = 0,$$

$$\Sigma P\,(x\cos\beta - y\cos\alpha) = 0.$$

The last three equations may be written thus

$$\frac{\Sigma Px}{\cos\alpha} = \frac{\Sigma Py}{\cos\beta} = \frac{\Sigma Pz}{\cos\gamma}.$$

Hence we can deduce the conditions that a system of *parallel* forces may maintain a body in equilibrium, *however they may be turned about their points of application.* For the preceding equations must then hold whatever α, β, γ may be. Thus we must have

$$\Sigma P = 0, \quad \Sigma Px = 0, \quad \Sigma Py = 0, \quad \Sigma Pz = 0.$$

We leave it as an exercise for the student to deduce from these equations that the forces can be divided into two equal and opposite sets which have the same centre.

79. In Art. 72 we have reduced the forces acting on a body to a force R and a couple G. If G vanish there remains a single force; and if R vanish, a single couple. If neither R nor G vanish the forces *may* reduce to a single force; we proceed to shew when this is possible.

To find the condition among the forces that they be reducible to a single force.

Any system of forces can be reduced to a single force R and a couple G; if then the forces can be reduced to a single

force S, it follows that G, R, and $-S$ are in equilibrium. If R and $-S$ do not form a couple, they can be reduced to a couple G' and a force R'; therefore R' must balance the couple compounded of G and G'. This is impossible (see Art. 73). Hence R and $-S$ must form a couple, and this couple must have its plane coincident with that of G, or parallel to that of G, in order that it may balance G. Therefore that the forces may reduce to a single force, the direction of R must be parallel to the plane of G, or coincident with it: that is, must be at *right angles to the axis of* G. Hence, using the notation of Art. 72,

$$\cos a \cos \lambda + \cos b \cos \mu + \cos c \cos \nu = 0,$$

therefore

$$L\Sigma X + M\Sigma Y + N\Sigma Z = 0.$$

80. Conversely, if $L\Sigma X + M\Sigma Y + N\Sigma Z = 0$, and ΣX, ΣY, ΣZ *do not all vanish*, the forces can be reduced to a single force. For the plane of the couple G may be made to contain the force R, and the couple may be supposed to have each of its forces $= R$ and its arm consequently $= \frac{G}{R}$; the couple may then be turned round in its own plane until the force at one end of its arm balances the resultant force R, and there remains R at the other end of the arm.

81. *When the forces are reducible to a single force, to find the equations to the straight line in which it acts.*

Let L, M, N denote the moments of the forces round the co-ordinate axes; L', M', N' the moments of the forces round axes parallel to the co-ordinate axes drawn through the point (x', y', z'). Then L' is found by writing $y_1 - y'$ for y_1, $y_2 - y'$ for y_2, ... $z_1 - z'$ for z_1, $z_2 - z'$ for z_2 ..... in the expression $\Sigma(Zy - Yz)$. Therefore

$$\begin{aligned} L' &= \Sigma\{Z(y - y') - Y(z - z')\} \\ &= L - y'\Sigma Z + z'\Sigma Y. \end{aligned}$$

Similarly

$$\begin{aligned} M' &= \Sigma\{X(z - z') - Z(x - x')\} \\ &= M - z'\Sigma X + x'\Sigma Z, \\ N' &= \Sigma\{Y(x - x') - X(y - y')\} \\ &= N - x'\Sigma Y + y'\Sigma X. \end{aligned}$$

If x', y', z' can be so taken as to make L', M', and N' vanish, the forces reduce to a single force passing through the point (x', y', z'). The three equations

$$L - y'\Sigma Z + z'\Sigma Y = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$M - z'\Sigma X + x'\Sigma Z = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

$$N - x'\Sigma Y + y'\Sigma X = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (3),$$

are equivalent to two independent equations; for if we eliminate z' from (1) and (2), we have

$$L\Sigma X + M\Sigma Y + \Sigma Z\,(x'\Sigma Y - y'\Sigma X) = 0,$$

which, combined with (3), will not determine x' and y', but leads to

$$L\Sigma X + M\Sigma Y + N\Sigma Z = 0$$

as the condition of possibility (compare Art. 79). If the forces satisfy this condition, the three planes represented by (1), (2) and (3) have a common line of intersection, at every point of which the resultant couple vanishes, and along which therefore the single resultant force acts.

82. The direction cosines of the normal to the plane represented by (1) are proportional to $0, -\Sigma Z, \Sigma Y$.

Let λ, μ, ν be the direction cosines of the line of intersection; then since this line is perpendicular to the normal to (1), we have

$$-\mu\Sigma Z + \nu\Sigma Y = 0, \quad \text{or} \quad \frac{\mu}{\Sigma Y} = \frac{\nu}{\Sigma Z}.$$

Its perpendicularity to the normal to (2) gives in like manner

$$\frac{\lambda}{\Sigma X} = \frac{\nu}{\Sigma Z}.$$

Thus the direction cosines of the line of action of the single resultant force are proportional to ΣX, ΣY, ΣZ, as might have been anticipated.

In order that the force which replaces the system may pass through the origin, we must have

$$L = 0, \quad M = 0, \quad N = 0.$$

83. Although a system of forces cannot always be reduced to a single force, it can always be reduced to *two* forces. For we have shewn that the system may be replaced by a force R at the origin, and a couple G lying in a plane through the origin; one of the forces of G may be supposed to act at the origin, and may be compounded with R, so that this resultant and the other force of G are equivalent to the whole system. Since the origin is arbitrary, we see that when a system of forces is not reducible to a single force it can be reduced to two forces, one of which can be made to pass through any assigned point.

84. *When three forces maintain a body in equilibrium, they must lie in the same plane.*

Draw any straight line intersecting the directions of two of the forces and not parallel to the third force, and take this straight line for the axis of x. Then the first two forces have no moment round the axis of x; therefore the equation $L = 0$ requires that the third force should have no moment round the axis of x; that is, the direction of the third force must pass through the axis of x. Since then *any* straight line, which meets the lines of action of two of the forces, and is not parallel to the third, meets the line of action of the third, the three forces must lie in one plane.

Combining this proposition with that in Art. 58, we see that if three forces keep a body in equilibrium, they must all lie in the same plane and must meet at a point or be parallel.

85. If the axes of co-ordinates be oblique, suppose l, m, n to denote the sines of the angles between the axes of y and z, z and x, x and y, respectively; then we may shew, as in Art. 72, that any system of forces can be reduced to ΣX, ΣY, ΣZ, acting at the origin along the axes of x, y, z respectively, and three couples in the three co-ordinate planes, having their moments equal to lL, mM, nN respectively, where, as before,

$$L = \Sigma (Zy - Yz), \quad M = \Sigma (Xz - Zx), \quad N = \Sigma (Yx - Xy).$$

Also for equilibrium, we must have, as before,

$$\Sigma X = 0, \quad \Sigma Y = 0, \quad \Sigma Z = 0;$$

$$L = 0, \quad M = 0, \quad N = 0.$$

That the forces may reduce to a single force we must have, as before,

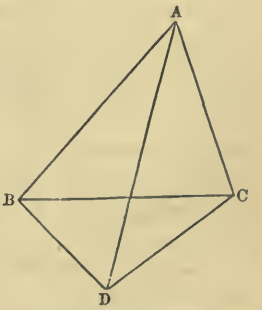
$$L\Sigma X + M\Sigma Y + N\Sigma Z = 0,$$

and ΣX, ΣY, ΣZ not all vanishing.

The following propositions are connected with the subject of the present Chapter.

I. Forces act at the angular points of a tetrahedron in directions respectively perpendicular to the opposite faces and proportional to the areas of the faces in magnitude: shew that the forces will be in equilibrium.

Let $ABCD$ represent the tetrahedron.



(1) Resolve the forces parallel to AB. Let p denote the perpendicular from A on the face BCD; then the resolved part of the force at A is $\frac{p}{AB} \times$ area of BCD, that is,

$$\frac{3 \text{ volume of tetrahedron}}{AB}.$$

We obtain the same expression for the resolved part of the force at B. The forces at C and D have no resolved part parallel to AB. Thus the forces resolved parallel to AB vanish.

(2) Take moments round AB. Let q denote the perpendicular from C on the straight line AB; θ the angle between the planes BAD and BAC. Then the moment of the force at C is $q \cos\theta$. area of ABD, that is,

$$\frac{qAB\cos\theta \text{ . area of } ABD}{AB} \text{ or } \frac{2\cos\theta \text{ . area of } ABC \text{ . area of } ABD}{AB}.$$

We obtain the same expression for the moment of the force at D. Thus the moments round AB vanish.

Since these results hold for *any* edge of the tetrahedron the forces must be in equilibrium.

II. Four forces act on a tetrahedron at right angles to the faces and proportional to their areas, the points of application of the forces being the centres of the circles circumscribing the faces: shew that if the forces all act inwards or all act outwards they will be in equilibrium.

In this case the forces all pass through a point, namely the centre of the sphere described round the tetrahedron. Hence we only require the first part of the investigation in the preceding proposition to establish that the forces are in equilibrium.

Or we may resolve the forces at right angles to a face instead of parallel to an edge, and thus obtain the result. For resolve the forces at right angles to the face *BCD*; we have one force represented by the area *BCD*, and the resolved parts of the other forces are represented by the projections of the respective areas *BAC*, *CAD*, *DAB* on *BCD*. And the sum of these projections is equal to *BCD*. Thus the forces resolved at right angles to *BCD* vanish.

Similarly the forces resolved at right angles to any other face vanish.

III. By a process similar to that used in establishing the Proposition I. at the end of Chapter IV. we can extend the preceding Proposition to the case of any polyhedron bounded by triangular faces. Thus we obtain the following result: Forces act on a polyhedron bounded by triangular faces at right angles to the faces and proportional to their areas, the points of application of the forces being the centres of the circles circumscribing the faces; shew that if the forces all act inwards or all act outwards they will be in equilibrium.

IV. If four forces acting on a rigid body are in equilibrium, and a tetrahedron be constructed by drawing planes at right angles to the directions of the forces, the forces will be respectively proportional to the areas of the faces.

This is the converse of II. and may be readily demonstrated: for by resolving the forces in any direction, and projecting the areas on a plane at right angles to that direction, we find that the four forces are connected by the same linear relation as the four areas.

We infer from this result that the areas in the present theorem must be respectively proportional to the volumes considered in the Proposition x. at the end of Chapter II.: thus we indirectly arrive at a geometrical truth.

V. Parallel forces P, Q, R, S, T act at the corners of a regular pentagon: find under what conditions their centre will coincide with that of the pentagon.

Let c stand for $2\cos 36^0$; then, by Art. 66, the following condition is necessary and sufficient in order that the centre of the parallel forces should fall on the straight line through the centre of the pentagon and the point at which P acts:

$$cQ + R = cT + S \quad \ldots\ldots(1).$$

Similarly, the following condition is necessary and sufficient in order that the centre of the parallel forces should fall on the straight line through the centre of the pentagon and the point at which Q acts:

$$cR + S = cP + T \quad \ldots\ldots(2).$$

Hence (1) and (2) are the necessary and sufficient conditions in order that the centre of the parallel forces may coincide with that of the pentagon.

We may observe that in the same manner as (1) and (2) were obtained we have the following as the necessary and sufficient condition that the centre of the parallel forces should fall on the straight line through the centre of the pentagon and the point at which R acts:

$$cS + T = cQ + P \quad \ldots\ldots(3).$$

If we eliminate R between (1) and (2) we obtain

$$c^2Q + cP = (c^2 - 1)\,T + (c+1)\,S \quad \ldots\ldots(4).$$

In order that (3) and (4) may agree we must have $c^2 - c - 1 = 0$. Thus we see that from statical considerations we can obtain an equation for determining the value of $\cos 36^0$.

EXAMPLES.

1. Four parallel forces act at the angles of a plane quadrilateral and are inversely proportional to the segments of its diagonals nearest to them; shew that the point of application of their resultant lies at the intersection of the diagonals.

2. Parallel forces act at the angles A, B, C of a triangle and are respectively proportional to a, b, c; shew that their resultant acts at the centre of the inscribed circle.

3. A cone whose vertical angle is 30°, and whose weight is W, is placed with its vertex on a smooth horizontal plane; shew that it may be kept with its slant side in a vertical position by a couple whose arm is equal to the length of the slant side of the cone, and each force $\frac{3W}{16}$.

4. Six equal forces act along the edges of a cube which do not meet a given diagonal, taken in order; find their resultant.

Result. A couple, the moment of which is $2Pa\sqrt{3}$, where P denotes each force and a the edge of the cube.

5. A cube is acted on by four forces; one force is in a diagonal, and the others in edges no two of which are in the same plane and which do not meet the diagonal; find the condition that the forces may have a single resultant.

Result. $(XY+YZ+ZX)\sqrt{3}+P(X+Y+Z)=0$; where X, Y, Z denote the forces along the edges, and P the force along the diagonal.

6. If a triangle is suspended from a fixed point by strings attached to the angles, the tension of each string is proportional to its length.

7. A uniform heavy triangle is supported in a horizontal position by three parallel strings attached to the three sides respectively; shew that there is an infinite number of ways in which the strings may be relatively disposed so that their tensions may be equal, but that the situation of one being given, that of each of the other two is determinate.

8. A sphere of given weight rests on three planes whose equations are

$$x\cos\alpha + y\cos\beta + z\cos\gamma = 0,$$
$$x\cos\alpha_1 + y\cos\beta_1 + z\cos\gamma_1 = 0,$$
$$x\cos\alpha_2 + y\cos\beta_2 + z\cos\gamma_2 = 0,$$

the axis of z being vertical; find the pressure on each plane.

9. A heavy triangle ABC is suspended from a point by three strings, mutually at right angles, attached to the angular points of the triangle; if θ be the inclination of the triangle to the horizon in its position of equilibrium, then

$$\cos\theta = \frac{3}{\sqrt{(1+\sec A\sec B\sec C)}}.$$

10. An equilateral triangle without weight has three unequal particles placed at its angular points; the system is suspended from a fixed point by three equal strings at right angles to each other fastened to the corners of the triangle; find the inclination of the plane of the triangle to the horizon.

Result. The cosine of the angle is $\dfrac{W_1+W_2+W_3}{\sqrt{\{3(W_1^2+W_2^2+W_3^2)\}}}$, where W_1, W_2, W_3 represent the weights of the particles.

11. Four smooth equal spheres are placed in a hemispherical bowl. The centres of three of them are in the same horizontal plane, and that of the other is above it. If the radius of each sphere be one-third that of the bowl, shew that the mutual pressures of the spheres are all equal; and find the pressure of each of the lower spheres on the bowl.

Results. Let W be the weight of each of the spheres; then each of the mutual pressures between the spheres is $\dfrac{W}{\sqrt{6}}$; and $\dfrac{4W}{\sqrt{6}}$ is the pressure of each of the lower spheres on the bowl.

12. Three equal spheres hang in contact from a fixed point by three equal strings; find the heaviest sphere of given radius that may be placed upon them without causing them to separate.

Result. Let W be the weight of each of the equal spheres, θ the angle which each string makes with the vertical, ϕ the angle which the line joining the centre of one of the three equal spheres with the centre of the upper sphere makes with the vertical; then the weight of the upper sphere must not exceed $\dfrac{3W\tan\theta}{\tan\phi - \tan\theta}$.

13. $ABCD$ is a tetrahedron in which the edges AB, AC, AD are at right angles to each other; forces are represented in magnitude and direction by AB, AC, AD, BC, CD, DB: determine their resultant.

14. Three equal hollow spheres rest symmetrically inside a smooth paraboloid of revolution, whose axis is vertical; a solid sphere of equal radius is placed upon them: shew that the equilibrium will be destroyed if the radius of the spheres is less than $\dfrac{l}{2\sqrt{6}}$, where l is the latus rectum; the weight of the hollow spheres being neglected in comparison with that of the solid one.

CHAPTER VI.

EQUILIBRIUM OF A CONSTRAINED BODY.

86. *To find the conditions of equilibrium of forces acting on a rigid body when one point is fixed.*

Let the fixed point be taken as the origin of co-ordinates. The action of the forces on the body will produce a pressure on the fixed point; let X', Y', Z' be the resolved parts of this pressure parallel to the axes. Then the fixed point will exert forces $-X'$, $-Y'$, $-Z'$ against the body; and if we take these forces in conjunction with the given forces, we may suppose the body to be free, and the equations of equilibrium are

$$\Sigma X - X' = 0, \quad \Sigma Y - Y' = 0, \quad \Sigma Z - Z' = 0,$$
$$L = 0, \qquad M = 0, \qquad N = 0.$$

The first three equations give the resolved parts of the pressure on the fixed point; and the last three are the only conditions to be satisfied by the given forces. Thus the forces will be in equilibrium if *the sums of the moments of the forces with respect to three straight lines at right angles to each other, and passing through the fixed point, vanish.*

Conversely, if the forces are in equilibrium the sum of the moments of the forces with respect to any straight line through the fixed point will vanish.

From the equations $X' = \Sigma X$, $Y' = \Sigma Y$, $Z' = \Sigma Z$, it follows that the pressure on the fixed point is equal to the resultant of all the given forces of the system moved parallel to themselves up to the fixed point.

If all the forces are parallel, we may take the axis of z passing through the fixed point parallel to the forces. Then all the forces included in ΣX vanish, and so do all the forces included in ΣY; thus N vanishes, M reduces to $-\Sigma Zx$, and L reduces to ΣZy. Hence X' and Y' vanish and the equations of equilibrium reduce to

$$\Sigma Z - Z' = 0, \quad \Sigma Zy = 0, \quad \Sigma Zx = 0;$$

the first determines the pressure on the fixed point, and the other two are conditions which must be satisfied by the given forces.

If all the forces act in one plane passing through the fixed point, and we take this plane for that of (x, y), all the forces included in ΣZ vanish; also the ordinate parallel to the axis of z of the point of application of each force is zero. Thus L and M vanish; also Z' vanishes, and the equations of equilibrium reduce to

$$\Sigma X - X' = 0, \quad \Sigma Y - Y' = 0, \quad \Sigma (Yx - Xy) = 0;$$

the first two determine the pressure on the fixed point, and the third is the only condition which the forces must satisfy. Thus the forces will be in equilibrium if *the sum of the moments of the forces with respect to the straight line perpendicular to their plane, and passing through the fixed point vanishes;* and conversely, if the forces are in equilibrium the sum of the moments of the forces with respect to this straight line will vanish.

87. *To find the condition of equilibrium of a body which has two points in it fixed.*

Let the axis of z pass through the two fixed points; and let the distances of the points from the origin be z' and z''. Also let X', Y', Z' be the resolved parts of the pressures on one point, and X'', Y'', Z'' those on the other point.

Then, as in Art. 86, the equations of equilibrium will be

$$\Sigma X - X' - X'' = 0, \quad \Sigma Y - Y' - Y'' = 0, \quad \Sigma Z - Z' - Z'' = 0,$$

$$L + Y'z' + Y''z'' = 0, \quad M - X'z' - X''z'' = 0, \quad N = 0.$$

The first, second, fourth, and fifth of these equations will determine X', X'', Y', Y''; the third equation gives $Z' + Z''$, shewing that the pressures on the fixed points in the direction of the line joining them are indeterminate, being connected by one equation only. The last is the only condition of equilibrium, namely $N = 0$. Thus the forces will be in equilibrium, if *the sum of the moments of the forces with respect to the straight line passing through the fixed points vanishes;* and conversely, if the forces are in equilibrium the sum of the moments of the forces with respect to this straight line will vanish.

88. The indeterminateness which occurs as to the values of Z' and Z'' might have been expected; for if two forces, $-Z'$ and $-Z''$, act on a rigid body *in the same straight line*, their effect will be the same at whatever point in their line of action we suppose them applied, and consequently they may be supposed both to act at the same point, or one of them to be increased provided the other be equally diminished. In practical cases, an additional datum is required to determine their separate values. For instance, the portions of the weight of a door borne by each of its two hinges will be greatly altered by a slight raising or lowering of one hinge.

The case which we have been considering is that of a body which is capable of turning round a *fixed axis;* for an axis will be fixed if two of its points are fixed.

89. If the body, instead of having two fixed points, can turn round an axis and *also slide along it*, then in addition to the condition $N=0$, we must have $\Sigma Z=0$, supposing the axis of z directed along the straight line on which the body can turn and slide. For the axis will not be able, as in the last case, to furnish any forces $-Z'$ and $-Z''$ to counteract ΣZ, and therefore ΣZ must $=0$.

90. *To find the conditions of equilibrium of a rigid body resting on a smooth plane.*

Let this plane be the plane of (x, y); and let x', y' be the co-ordinates of one of the points of contact, R' the pressure which the body exerts against the plane at that point. Then the force $-R'$, and similar forces for the other points of contact, taken in conjunction with the given forces, ought to satisfy the equations of equilibrium; hence

$$\Sigma X=0,\quad \Sigma Y=0,\quad \Sigma Z-R'-R''-\ldots=0,$$

$$L-R'y'-R''y''-\ldots=0,\quad M+R'x'+R''x''+\ldots=0,\quad N=0.$$

If only one point be in contact with the plane, then the third equation gives the pressure, and we have five equations of condition,

$$\Sigma X=0,\quad \Sigma Y=0,\quad L-y'\Sigma Z=0,\quad M+x'\Sigma Z=0,\quad N=0.$$

If two points be in contact, then the equations

$$R'y' + R''y'' = L, \quad R'x' + R''x'' = -M,$$

give $$R' = \frac{Lx'' + My''}{y'x'' - x'y''}, \quad R'' = -\frac{Lx' + My'}{y'x'' - x'y''};$$

and the equations of condition are

$$\Sigma X = 0, \ \Sigma Y = 0, \ \Sigma Z - \frac{L(x'' - x') + M(y'' - y')}{y'x'' - x'y''} = 0, \text{ and } N = 0.$$

If three points are in contact, then the pressures are determined from the equations

$$R' + R'' + R''' = \Sigma Z,$$
$$R'y' + R''y'' + R'''y''' = L,$$
$$R'x' + R''x'' + R'''x''' = -M,$$

and the conditions of equilibrium are

$$\Sigma X = 0, \quad \Sigma Y = 0, \quad N = 0.$$

If more than three points are in contact, then the pressures are indeterminate, since they are connected by only three equations; but the conditions of equilibrium are still

$$\Sigma X = 0, \quad \Sigma Y = 0, \quad N = 0.$$

91. The equations at the commencement of the preceding Article shew that if a body rest in equilibrium against a plane, the forces which press it against the plane must reduce to a *single force* acting in a direction perpendicular to the plane, for the condition

$$L\Sigma X + M\Sigma Y + N\Sigma Z = 0$$

is satisfied, since ΣX, ΣY, and N vanish. Hence the forces reduce to a single force; and since ΣX and ΣY vanish, this force must be perpendicular to the fixed plane.

Also, this single force must counterbalance the forces $-R'$, $-R''$..., which are all parallel and all act in the same direction. Hence, from considering the construction given in Art. 66 for determining the centre of a system of parallel forces, it follows that the point where this resultant cuts the plane must be within a polygon, formed by so joining the points of contact as to include them all and to have no *re-entering angle.*

MISCELLANEOUS EXAMPLES.

1. The lid $ABCD$ of a cubical box, moveable about hinges at A and B, is held at a given angle to the horizon by a horizontal string connecting C with a point vertically over A: find the pressure on each hinge.

2. Two equal forces act on a cube whose centre is fixed, along diagonals which do not meet of two adjacent faces: find the couple which will keep the cube at rest.

Result. Let P denote each force, a the edge of the cube; the moment of the required couple is either $\frac{Pa\sqrt{3}}{2}$ or $\frac{Pa}{2}$ according to the directions of the two given forces.

3. Three equal heavy rods in the position of the three edges of an inverted triangular pyramid are in equilibrium under the following circumstances: their upper extremities are connected by strings of equal lengths, and their lower extremities are attached to a hinge about which the rods may move freely in all directions. Find the tension of the strings.

4. A given number of uniform heavy rods, all of the same weight, have their extremities jointed together at a common hinge, about which they can turn freely; and being introduced through a circular hole in a horizontal plane with their hinge end downwards, are spread out symmetrically along the circumference of the hole like the ribs of a conical basket. If a heavy sphere be now placed in the interior of the system of rods, so as to be supported by them, determine the position of rest.

5. A cylinder with its base resting against a smooth vertical plane is held up by a string fastened to it at a point of its curved surface whose distance from the vertical plane is h. Shew that h must be greater than $b - 2a \tan\theta$ and less than b, where $2b$ is the altitude of the cylinder, a the radius of the base, and θ the angle which the string makes with the vertical.

6. A cylinder rests with its base on a smooth inclined plane; a string attached to its highest point, passing over a pully at the top of the inclined plane, hangs vertically and supports a weight; the portion of the string between the cylinder and the pully is horizontal. Determine the conditions of equilibrium.

Results. Let W be the weight of the cylinder, W' the weight attached to the string, α the inclination of the plane to the horizon; then $W' = W \tan \alpha$, and $\tan \alpha$ must not exceed the ratio of the diameter of the base of the cylinder to the height of the cylinder.

7. A cone of given weight W is placed with its base on an inclined plane, and supported by a weight W' which hangs by a string fastened to the vertex of the cone and passing over a pully in the inclined plane at the same height as the vertex. Determine the conditions of equilibrium.

Results. Let α be the inclination of the plane to the horizon, θ the semi-vertical angle of the cone; then

$W' = W \tan \alpha$, and $\tan \theta$ must not be less than $\frac{3}{8} \sin 2\alpha$.

8. A smooth hemispherical shell whose base is closed includes two equal spheres whose radii are one third of that of the shell. The shell is fixed with its base vertical; find the mutual pressures at all the points of contact.

Results. Let R_1 be the pressure between the upper sphere and the shell, R_2 that between the two spheres, R_3 that between the lower sphere and the base of the shell, R_4 that between the lower sphere and the curved part of the shell; then

$$R_1 = \frac{W}{\sqrt{3}}, \quad R_2 = \frac{2W}{\sqrt{3}}, \quad R_3 = \frac{3W}{\sqrt{3}}, \quad R_4 = \frac{4W}{\sqrt{3}}.$$

9. A rectangular table is supported in a horizontal position by four legs at its four angles: a given weight W being placed upon a given point of it, shew that the pressure on each leg is indeterminate, and find the greatest and least value it can have for a given position of the weight.

CHAPTER VII.

GENERAL THEOREMS ON A SYSTEM OF FORCES.

92. In Art. 72 it is proved that the forces acting on a rigid body may be reduced to a force R acting at a point chosen at pleasure, and a couple G. In the present chapter we shall call the components of R along the three axes X, Y, Z instead of ΣX, ΣY, ΣZ. The components of G will still be called L, M, N.

We have $X^2+Y^2+Z^2=R^2$, and $L^2+M^2+N^2=G^2$. The direction cosines of R are $\frac{X}{R}, \frac{Y}{R}, \frac{Z}{R}$; and the direction cosines of the axis of G are $\frac{L}{G}, \frac{M}{G}, \frac{N}{G}$. Hence if θ denote the inclination of these two lines, we have

$$XL+YM+ZN=RG\cos\theta.$$

93. Resolve G into two component couples, having their axes respectively parallel and perpendicular to the force R. The component with its axis parallel to R and therefore with its plane perpendicular to R is $G\cos\theta$, that is

$$\frac{XL+YM+ZN}{R}.$$

The other component $G\sin\theta$ has its axis perpendicular to R and its plane therefore parallel to R. We may carry it till its plane contains R, and it will then, by Art. 59, compound with R into a single force equal to R, at a distance $\frac{G\sin\theta}{R}$ from the original line of action of R. We have thus reduced the system to a single force R with this new line of action, and a couple $G\cos\theta$ in a plane perpendicular to R.

It is therefore always possible to reduce a system of forces to a single force and a couple whose plane is perpendicular to the force. The line of action of this force (which is also the direction of the axis of the accompanying couple) is called the *Central Axis* of the system. In special cases either the force along the central axis or the couple round it may vanish. If the force vanishes, the system reduces to a couple, and the central axis becomes indeterminate in position, though still fixed in direction. If the couple vanishes, the system reduces to a single force and the central axis is its line of action.

94. Since the couple round the central axis is a definite thing, its value $\frac{XL+YM+ZN}{R}$ will be the same whatever origin and axes of reference we employ. R is also definite, and hence their product

$$XL+YM+ZN$$

is also independent of the choice of origin and axes. If this product vanishes while R is finite, the couple vanishes and the system reduces to a single force. Conversely, if the system is equivalent to a single force $XL+YM+ZN$ must vanish.

95. If the origin is on the central axis, the directions of R and G are the same; hence, by comparison of their direction cosines, we have

$$\frac{X}{L}=\frac{Y}{M}=\frac{Z}{N},$$

and conversely if these equations are satisfied the origin is on the central axis.

When the origin is not on the central axis, the angle θ will not be zero nor 180°, and $G\cos\theta$ will be arithmetically less than G. Hence the value of G is least when the origin is on the central axis. The couple $G\cos\theta$ is therefore often called the *minimum couple* of the system.

96. If the origin be taken at a distance h from the central axis, the couple G will be compounded of the minimum couple (whose plane is perpendicular to the central axis) and

a couple Rh whose plane contains the central axis. If we denote the minimum couple by G_0, we shall have

$$G^2 = G_0^2 + R^2h^2.$$

G will accordingly have the same numerical value for all origins which are at the same distance from the central axis.

The plane of the couple Rh which has been compounded with the minimum couple is the plane drawn through the central axis and the origin, and is the same for all origins situated on the same line parallel to the central axis. In origins so situated the couple G will be the same not only in magnitude but in the direction of its axis.

As regards the physical meaning of the couple G, it denotes a couple equal and opposite to that which would suffice to keep the body from turning round the origin as a fixed point: it may be called the *resultant couple round the origin.*

97. To know in what direction the central axis lies from a given origin when R and G are given both in magnitude and direction, it is to be remembered that in compounding R with the couple $G \sin\theta$ the shifting of R takes place in the plane of the couple $G \sin\theta$. This is the plane drawn through R at right angles to the plane of the angle θ, that is, to the plane which contains R and the axis of G. We shall therefore arrive at the central axis by travelling from the origin in a direction perpendicular both to R and the axis of G, through a distance $\dfrac{G \sin\theta}{R}$.

Suppose for example that R acts along Ox, and that the axis of G is directed between Ox and Oy in the plane of (x, y). The axis of the couple $G \sin\theta$ will be directed along Oy, and the couple will tend to turn from Oz towards Ox; hence when one of its two forces is placed opposite to R at O, the other must act at a point in Oz at which z is positive. The central axis will therefore be a line drawn parallel to Ox through a point on the positive axis of z.

98. In general, to find the co-ordinates a, b, c of the foot of the perpendicular from the origin on the central axis, we

have the three equations

$$aX + bY + cZ = 0,$$
$$aL + bM + cN = 0,$$
$$a^2 + b^2 + c^2 = \frac{G^2 \sin^2\theta}{R^2},$$

the first expressing that the perpendicular is at right angles to R, the second that it is at right angles to G, and the third giving its length.

The first two give

$$\frac{a}{YN - ZM} = \frac{b}{ZL - XN} = \frac{c}{XM - YL} = k \text{ suppose,}$$

and by substituting for a, b, c in the third we have

$$k^2\{(YN - ZM)^2 + (ZL - XN)^2 + (XM - YN)^2\} = \frac{G^2 \sin^2\theta}{R^2}.$$

that is (see Art. 77), $k^2 R^2 G^2 \sin^2\theta = \frac{G^2 \sin^2\theta}{R^2}$,

whence $$k^2 = \frac{1}{R^4}.$$

This leaves a doubt as to whether k is $\frac{1}{R^2}$ or $-\frac{1}{R^2}$. By examining the particular case considered at the end of the preceding article, it will be found that the positive sign must be taken, the required co-ordinates are therefore

$$a = \frac{YN - ZM}{R^2}, \quad b = \frac{ZL - XN}{R^2}, \quad c = \frac{XM - YL}{R^2}.$$

We shall now confirm this determination of the value of k by an independent investigation.

99. It has been shewn in Art. 81 that the three component couples round the point (x, y, z) are

$$L - yZ + zY, \quad M - zX + xZ, \quad N - xY + yX.$$

The condition of the point (x, y, z) being on the central axis is, by Art. 95, that these components are proportional to

X, Y, Z. Hence the equations to the central axis may be written

$$\frac{L - yZ + zY}{X} = \frac{M - zX + xZ}{Y} = \frac{N - xY + yX}{Z}.$$

Multiply the numerator and denominator of the first member by X, of the second by Y, and of the third by Z, and then take the sums of numerators and of denominators. We thus obtain

$$\frac{L - yZ + zY}{X} = \frac{XL + YM + ZN}{X^2 + Y^2 + Z^2},$$

that is, $L(Y^2 + Z^2) - (yZ - zY)R^2 = X(YM + ZN)$,

$$Y(zR^2 - XM + YL) = Z(yR^2 - ZL + XN),$$

and dividing by YZR^2,

$$\frac{1}{Z}\left(z - \frac{XM - YL}{R^2}\right) = \frac{1}{Y}\left(y - \frac{ZL - XN}{R^2}\right) = \frac{1}{X}\left(x - \frac{YN - ZM}{R^2}\right).$$

Hence the central axis is a line parallel to R passing through the point whose co-ordinates are

$$a = \frac{YN - ZM}{R^2}, \quad b = \frac{ZL - XN}{R^2}, \quad c = \frac{XM - YL}{R^2}.$$

100. When a given system of forces acting on a rigid body is reduced to two forces, and these are represented by two straight lines, the volume of the tetrahedron of which the two straight lines are opposite edges is constant.

Let AP and BQ be the two straight lines representing the forces P and Q. The tetrahedron of which they are opposite edges can easily be shewn to be one-sixth of a parallelepiped of which the edges are respectively equal and parallel to AP, BQ, and AB. The volume of this parallelepiped is not affected by sliding AP and BQ along the lines of action of the two forces, for this change leaves unaltered the parallelogram under AP and BQ which is the base of the parallelepiped, and also leaves the altitude unchanged since this is the common perpendicular of the two lines of action. We shall therefore suppose AP and BQ to be so placed that AB is the common perpendicular.

We shall shew that the volume of the parallelepiped is equal to the product of the resultant force R and the minimum couple.

Take B as origin, then R is represented by the diagonal through B of the parallelogram whose sides are BQ and a line equal and parallel to AP. The whole couple is $P \,.\, AB$, and by resolving P parallel and perpendicular to R we obtain the two couples $P\cos(P, R)\, AB$ which lies in a plane through R, and $P\sin(P, R)\, AB$ which is in a plane perpendicular to R and is the minimum couple. The product of R and the minimum couple is therefore

$$R \,.\, P\sin(P, R)\, AB.$$

But $R \,.\, P\sin(P, R)$ is the area of the base of the parallelepiped and AB is its altitude. Therefore the product of the resultant force and minimum couple is equal to the volume of the parallelepiped under AP, BQ, AB, or to six times the tetrahedron under AP, BQ.

Since $R\sin(P, R)$ is equal to $Q\sin(P, Q)$, another expression for the volume of the parallelepiped is

$$P \,.\, Q\sin(P, Q)\, AB,$$

that is, it is equal to either of the two forces multiplied by the moment of the other force round it.

101. Remove the restriction as to the position of the two points A, B, and let them be anywhere on the lines of action of the two forces. Let the co-ordinates of A be $x_1 y_1 z_1$ and those of B, $x_2 y_2 z_2$. Then the projections of BA on the axes of co-ordinates are $x_1 - x_2$, $y_1 - y_2$, $z_1 - z_2$, the projections of AP are X_1, Y_1, Z_1, and those of BQ, X_2, Y_2, Z_2. Hence the volume of the parallelepiped is

$$\begin{vmatrix} x_1 - x_2, & X_1, & X_2 \\ y_1 - y_2, & Y_1, & Y_2 \\ z_1 - z_2, & Z_1, & Z_2 \end{vmatrix}$$

and the tetrahedron is one-sixth of this.

The above determinant is accordingly equal to the value of

$$XL + YM + ZN$$

for the system composed of the two forces AP, BQ. This may be verified as follows.

We have

$$X = X_1 + X_2, \quad L = L_1 + L_2 = y_1 Z_1 - z_1 Y_1 + y_2 Z_2 - z_1 Y_2$$
$$= \begin{vmatrix} y_1, & Y_1 \\ z_1, & Z_1 \end{vmatrix} + \begin{vmatrix} y_2, & Y_2 \\ z_2, & Z_2 \end{vmatrix}.$$

Thus $XL + YM + ZN$ is the sum of the four determinants

$$\begin{vmatrix} X_1, x_1, X_1 \\ Y_1, y_1, Y_1 \\ Z_1, z_1, Z_1 \end{vmatrix} + \begin{vmatrix} X_1, x_2, X_2 \\ Y_1, y_2, Y_2 \\ Z_1, z_2, Z_2 \end{vmatrix} + \begin{vmatrix} X_2, x_1, X_1 \\ Y_2, y_1, Y_1 \\ Z_2, z_1, Z_1 \end{vmatrix} + \begin{vmatrix} X_2, x_2, X_2 \\ Y_2, y_2, Y_2 \\ Z_2, z_2, Z_2 \end{vmatrix}$$

of which the first and last vanish identically, and the sum of the remaining two may be written

$$\begin{vmatrix} x_1, & X_1, & X_2 \\ y_1, & Y_1, & Y_2 \\ z_1, & Z_1, & Z_2 \end{vmatrix} - \begin{vmatrix} x_2, & X_1, & X_2 \\ y_2, & Y_1, & Y_2 \\ z_2, & Z_1, & Z_2 \end{vmatrix} = \begin{vmatrix} x_1 - x_2, & X_1, & X_2 \\ y_1 - y_2, & Y_1, & Y_2 \\ z_1 - z_2, & Z_1, & Z_2 \end{vmatrix}.$$

This last determinant is accordingly equal to the product of the resultant force and minimum couple of the system composed of any two given forces; $X_1 Y_1 Z_1$, $X_2 Y_2 Z_2$ being the components of the two forces, and $x_1 y_1 z_1$, $x_2 y_2 z_2$ their points of application.

102. In a system of n forces, $\frac{1}{2}n(n-1)$ different pairs can be selected. Let the product of the resultant force and minimum couple be formed for each pair. We shall now shew that the sum of these products is the product of the resultant force and minimum couple for the whole system.

For the whole system we have $X = X_1 + X_2 + \ldots + X_n$, with similar expressions for Y and Z. Also $L = L_1 + L_2 + \ldots + L_n$, with similar values for M and N. The product of the resultant force and minimum couple for the whole system is

$$XL + YM + ZN,$$

and when the above values are substituted XL becomes

$$X_1 L_1 + X_1 L_2 + X_2 L_1 + X_2 L_2 + \&c.$$

Expressing YM and ZN in like manner, we have

$$XL + YM + ZN$$
$$= \Sigma (X_1 L_1 + Y_1 M_1 + Z_1 N_1) + \Sigma (X_1 L_2 + Y_1 M_2 + Z_1 N_2 + X_2 L_1 + Y_2 M_1 + Z_2 M_1).$$

Again, the sum of products of resultant force and minimum couple for each pair is

$$(X_1+X_2)(L_1+L_2)+(Y_1+Y_2)(M_1+M_2)+(Z_1+Z_2)(N_1+N_2)$$
$$+(X_1+X_3)(L_1+L_3)+\&c.$$
$$=(n-1)\Sigma(X_1L_1+Y_1M_1+Z_1N_1)$$
$$+\Sigma(X_1L_2+Y_1M_2+Z_1N_2+X_2L_1+Y_2M_1+Z_2N_1).$$

This differs from the foregoing expression by

$$(n-2)\Sigma(X_1L_1+Y_1M_1+Z_1N_1),$$

which is zero, since each trio $X_1L_1+Y_1M_1+Z_1N_1$ is zero as has been shewn in the preceding article, and as may otherwise be inferred from the consideration that it is the product of the resultant force and minimum couple of a system consisting of a single force. (See Art. 94.)

The rule of signs for the product of a resultant force and minimum couple is, that it is positive when the force and the axis of the couple have the same direction, negative when they have opposite directions. That this rule is consistent with the above demonstration appears from the equation (for any system of forces)

$$XL+YM+ZN=RG\cos(R,G),$$

which shews that the sign of the first member is the same as that of $\cos(R,G)$.

The above result may also be stated in the two following forms (see Art. 100).

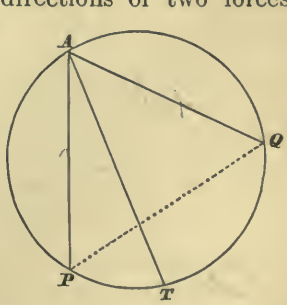
The algebraic sum of the $\frac{1}{2}n(n-1)$ products obtained by multiplying one force of each pair by the moment of the other force round it, is equal to the single product thus obtained from the two forces to which the system can be reduced.

Or again; If the forces of a system are represented by straight lines, the algebraic sum of the tetrahedrons obtained by taking each pair of lines as opposite edges is equal to the single tetrahedron similarly obtained from the two forces to which the system can be reduced.

The rule of signs in both cases is, that when the rotation which one force of a pair tends to produce round the other has to the direction of that other the relation which the rotation of a right-handed screw has to the direction of advance, one sign must be given; and when the relation agrees with a left-handed screw the opposite sign must be given.

103. When a system of *parallel* forces acting on a rigid body at given points of application reduces to a single force, that force always passes through a fixed point in the body whatever may be the position of the body. When *any* system of forces acts on a rigid body we might investigate the consequences of turning the body from one position into another while the forces retain their original directions, or of turning the forces in such a manner as to leave their *relative* directions unchanged while the body remains fixed. We shall here give some examples of the general theorems that have been demonstrated on this subject. The forces are supposed to act at fixed points in the body.

104. Let PA and QA be the directions of two forces lying in one plane, acting at the points P and Q respectively; TA the direction of their resultant. Suppose the forces in PA, QA to be turned round the points P and Q respectively through the same angle α towards the same direction; since PA and QA will include the same angle as before, their point of intersection will move on a circle passing through P and Q. And as the magnitudes of the forces are supposed unchanged, the magnitude of the resultant and the angles which it makes with the components remain unchanged. Hence if T be the intersection of the resultant and the circle originally, it will always be so, since the arcs PT and QT are proportional to the angles PAT and QAT; the resultant will therefore have turned through the angle α round the point T.



The same conclusion holds if instead of supposing the body

to be fixed and the forces to revolve, we suppose each force to remain parallel to itself and the body to be turned through any angle round a perpendicular to the plane of the forces.

The point T through which the resultant always passes is called the *astatic centre* of the forces which act at P and Q. It is evident, in like manner, that if a third force pass through a fixed point S and meet the straight line TA, we may find the astatic centre of the forces at T and S, that is, of the forces at P, Q, and S; and generally we may infer that every system of forces in one plane, when the points of application of the forces are given, has an *astatic centre*, unless the forces are in equilibrium or reducible to a couple.

105. If the origin is the astatic centre of forces P_1, P_2,...P_n applied at the points (x_1, y_1), (x_2, y_2),... and inclined at given angles α_1, α_2,... α_n to a fixed line in the body, (which we shall take as the axis of x,) their resultant will still pass through the origin when their inclinations are changed to $\alpha_1 + \theta$, $\alpha_2 + \theta$,.... Hence, by taking moments round the origin, we have

$$P_1\{x_1 \sin(\alpha_1+\theta) - y_1 \cos(\alpha_1+\theta)\} + P_2\{x_2 \sin(\alpha_2+\theta) - y_2 \cos(\alpha_2+\theta)\} + \&c. = 0,$$

that is,

$$\cos\theta\{P_1x_1 \sin\alpha_1 - P_1y_1 \cos\alpha_1 + \ldots\} + \sin\theta\{P_1x_1 \cos\alpha_1 + P_1y_1 \sin\alpha_1 + \ldots\} = 0.$$

Since this is to be true for all values of θ, the coefficients of $\cos\theta$ and $\sin\theta$ must each vanish. Thus we have

$$\Sigma(Yx - Xy) = 0, \qquad \Sigma(Xx + Yy) = 0.$$

In general, if a, b are the co-ordinates of the astatic centre, we must have

$$\Sigma\{Y(x-a) - X(y-b)\} = 0, \quad \Sigma\{X(x-a) + Y(y-b)\} = 0,$$

that is, $\Sigma(Yx - Xy) = Ya - Xb$, $\Sigma(Xx + Yy) = Xa + Yb$.

Putting G for $\Sigma(Yx - Xy)$ which is the moment of the forces round the origin, and Γ for $\Sigma(Xx + Yy)$, these equations give as the co-ordinates of the astatic centre

$$a = \frac{YG + X\Gamma}{R^2}, \qquad b = \frac{Y\Gamma - XG}{R^2}.$$

If we take the axis of x parallel to the original direction of the resultant force R, we have $X = R$, $Y = 0$,

$$a = \frac{\Gamma}{R}, \qquad b = -\frac{G}{R}.$$

106. A system of forces acts on a rigid body: determine the conditions which must hold in order that when the system is resolved parallel to any straight line these resolved parts may be in equilibrium.

Take a straight line whose direction cosines are l, m, n. In order that the resolved parts of the forces parallel to this straight line may be in equilibrium we must have, by Art. 78,

$$\Sigma (lX + mY + nZ) = 0,$$

$$\frac{\Sigma (lX + mY + nZ)\, x}{l} = \frac{\Sigma (lX + mY + nZ)\, y}{m}$$

$$= \frac{\Sigma (lX + mY + nZ)\, z}{n}.$$

And as these are to be true for all ratios of l, m, n we must have

$$\Sigma X = 0, \quad \Sigma Y = 0, \quad \Sigma Z = 0,$$

$$\Sigma Xy = 0, \ \Sigma Xz = 0, \ \Sigma Yx = 0, \ \Sigma Yz = 0, \ \Sigma Zx = 0, \ \Sigma Zy = 0,$$

$$\Sigma Xx = \Sigma Yy = \Sigma Zz.$$

These are the necessary and sufficient conditions.

107. We have remarked in Art. 9 that the property of the divisibility of matter leads us to the supposition that every body consists of an assemblage of material particles or molecules which are held together by their mutual attraction. Now we are totally unacquainted with the nature of these molecular forces; if, however, we assume the two hypotheses that the action of any two molecules on each other is the same, and also that its direction is the straight line joining them, then we shall be able to deduce the conditions of equilibrium of a rigid body from those of a single particle.

To deduce the conditions of equilibrium of a rigid body from those of a single particle.

Let the body be referred to three rectangular axes; and let x_1, y_1, z_1 be the co-ordinates of one of its constituent particles; X_1, Y_1, Z_1 the resolved parts, parallel to the axes, of the forces which act on this particle exclusive of the molecular forces; P_1, P_2, P_3, ... the molecular forces acting on this particle; α_1, β_1, γ_1; α_2, β_2, γ_2; ..... the angles their respective directions make with the three axes of co-ordinates. Then, since this particle is held in equilibrium by the above forces, we have, by Art. 27,

$$X_1 + P_1 \cos \alpha_1 + P_2 \cos \alpha_2 + \ldots\ldots = 0 \ldots\ldots(1),$$

$$Y_1 + P_1 \cos \beta_1 + P_2 \cos \beta_2 + \ldots\ldots = 0 \ldots\ldots(2),$$

$$Z_1 + P_1 \cos \gamma_1 + P_2 \cos \gamma_2 + \ldots\ldots = 0 \ldots\ldots(3).$$

We shall have a similar system of equations for each particle in the body; if there be n particles there will be $3n$ equations. These $3n$ equations will be connected one with another, since any molecular force which enters into one system of equations must enter into a second system; this is in consequence of the *mutual* action of the particles.

There are two considerations which will enable us to deduce from these $3n$ equations *six* equations of condition, independent of the molecular forces. These will be the equations which the other forces must satisfy, in order that equilibrium may be maintained.

The first consideration is this, that the molecular actions are *mutual;* and that, consequently, if $P_1 \cos \alpha_1$ represent the resolved part parallel to the axis of x of any one of the molecular forces involved in the $3n$ equations, we shall likewise meet with the term $-P_1 \cos \alpha_1$ in another of those equations which have reference to the axis of x. Consequently, if we add all those equations together which have reference to the same axis, we have the three following equations of condition *independent of the molecular forces,*

$$\Sigma X = 0, \quad \Sigma Y = 0, \quad \Sigma Z = 0.$$

The second consideration is this: that the straight lines joining the different particles are the directions in which the molecular forces act.

Thus, let P_1 be the molecular action between the particles whose co-ordinates are (x_1, y_1, z_1) and (x_2, y_2, z_2),

$$P_1 \cos \alpha_1, \quad P_1 \cos \beta_1, \quad P_1 \cos \gamma_1,$$
$$-P_1 \cos \alpha_1, \quad -P_1 \cos \beta_1, \quad -P_1 \cos \gamma_1,$$

the corresponding resolved parts of P_1 for the two particles. Then

$$\cos \alpha_1 = \frac{x_2 - x_1}{r}, \quad \cos \beta_1 = \frac{y_2 - y_1}{r}, \quad \cos \gamma_1 = \frac{z_2 - z_1}{r},$$

where $r = \sqrt{\{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\}}$.

These enable us to obtain three more equations free from molecular forces; for if we multiply (1) and (2) by y_1 and x_1 respectively, and then subtract, we have

$$Y_1 x_1 - X_1 y_1 + \ldots + P_1 \{x_1 \cos \beta_1 - y_1 \cos \alpha_1\} + \ldots = 0 \ldots (4).$$

By the same process we obtain from the system of equations which refer to the particle (x_2, y_2, z_2),

$$Y_2 x_2 - X_2 y_2 + \ldots - P_1 \{x_2 \cos \beta_1 - y_2 \cos \alpha_1\} + \ldots = 0 \ldots (5).$$

But the values of $\cos \alpha_1$ and $\cos \beta_1$ given above lead to the condition

$$(x_2 - x_1) \cos \beta_1 - (y_2 - y_1) \cos \alpha_1 = 0.$$

Wherefore the equation

$$Y_1 x_1 - X_1 y_1 + \ldots\ldots + Y_2 x_2 - X_2 y_2 + \ldots\ldots = 0$$

will not involve P_1, the molecular action between the particles whose co-ordinates are x_1, y_1, z_1 and x_2, y_2, z_2 respectively.

It follows readily from what we have shewn, that if we form all the equations similar to (4) and (5), and add them together, we shall have a final equation

$$\Sigma (Yx - Xy) = 0,$$

independent of the molecular forces.

In like manner we should obtain

$$\Sigma (Zy - Yz) = 0, \quad \Sigma (Xz - Zx) = 0.$$

Moreover we can shew that these six equations are the *only* equations free from the molecular forces, supposing the body to be rigid, and consequently the molecules to retain their mutual distances invariable. For if a body consist of three molecules, there must evidently be three independent molecular forces to keep them invariable; if to these three molecules a fourth be added, we must introduce three new forces to hold it to the others; if we add a fifth molecule we must introduce three forces to hold this invariably to any three of those which are already rigidly connected; and so on; from which we see that there must be at least $3+3(n-3)$ or $3n-6$ forces. Hence the $3n$ equations resembling (1), (2), and (3) contain at least $3n-6$ independent quantities to be eliminated; and therefore there cannot be more than *six* equations of condition connecting the external forces and the co-ordinates of their points of application.

MISCELLANEOUS EXAMPLES.

1. Determine the central axis when there are two forces P and Q whose lines of action are defined by $z=c$, $y=x\tan\alpha$, and $z=-c$, $y=-x\tan\alpha$ respectively.

2. If P and Q are two forces whose directions are at right angles, shew that the distances of the central axis from their lines of action are inversely as P^2 to Q^2.

3. If $L=mX$, $M=nY$, $N=pZ$ (with the notation of Art. 92), shew that the co-ordinates of the foot of the perpendicular on the central axis are

$$\frac{(p-n)\,YZ}{R^2},\quad \frac{(m-p)\,ZX}{R^2},\quad \frac{(n-m)\,XY}{R^2};$$

and that the couple round the central axis is equal to the force along it multiplied by $\dfrac{mX^2+nY^2+pZ^2}{X^2+Y^2+Z^2}$.

4. If $L = mX$, $M = nY$, $Z = 0$, $N = 0$, shew that the central axis is a line parallel to the plane of (x, y) cutting the axis of z at a point $z = \dfrac{(n-m)\,XY}{X^2+Y^2}$, and inclined to the axis of x at an angle whose tangent is $\dfrac{Y}{X}$. Shew that, when all possible values are assigned to the ratio $Y : X$, the central axis makes one complete revolution parallel to the plane of (x, y), and that its distance varies from $\frac{1}{2}(n \sim m)$ on one side of this plane to the same distance on the other side. [The ruled surface thus generated is called the *cylindroid*.] Shew that the couple round the central axis is equal to the force along it multiplied by $\dfrac{mX^2+nY^2}{X^2+Y^2}$.

5. Forces act at the angular points of a tetrahedron in directions respectively perpendicular to the opposite faces, and proportional to the areas of the faces in magnitude: shew that the forces have the property considered in Art. 106.

6. Shew that within a quadrilateral there is but one point, at which forces acting towards the corners and proportional to the distances of the point from them, can be in equilibrium.

7. Two forces acting at a point are represented in magnitude and direction by straight lines drawn from that point: their sum is constant and their resultant is constant both in magnitude and direction. Find the locus of the extremities of the straight lines which represent the forces.

8. If forces P, Q, R acting at the centre O of a circular lamina along the radii OA, OB, OC be equivalent to forces P', Q', R' acting along the sides BC, CA, AB of the inscribed triangle, shew that

$$\frac{P \cdot P'}{BC} + \frac{Q \cdot Q'}{CA} + \frac{R \cdot R'}{AB} = 0.$$

9. A uniform rigid rod, of length $2a$, can turn in a horizontal plane about its middle point. At one end a string is tied which passes over a fixed pully, vertically above that

end, and at a distance b from it, and is then fastened to a given weight. The rod is then turned through an angle θ, and kept at rest in that position by a horizontal force P perpendicular to the rod through its other end. Prove that P will be a maximum if

$$\tan^4 \frac{\theta}{2} = \frac{b^2}{b^2 + 4a^2}.$$

10. Prove that a system of forces can be reduced in an infinite number of ways to a pair of equal forces, whose directions make any assigned angle with one another; and find the distance between these forces when the angle is given.

CHAPTER VIII.

CENTRE OF GRAVITY.

108. By *weight* we mean the force with which a body tends towards the earth.

It appears from experiment that the weight of a given body is invariable so long as the body remains at the same place on the earth's surface, but changes when the body is taken to a different place. We shall suppose therefore when we speak of the weight of a body that the body remains at one place.

When a body is such that the weight of any portion of it is proportional to the volume of that portion it is said to be of *uniform density;* for the purposes of Statics, the density of such a body may be measured by the ratio which the weight of any volume of it bears to the weight of an equal volume of some arbitrarily chosen body of uniform density.

The product of the density of a body into its volume is called its *mass.*

When a body is not of uniform density its density at any point is measured thus: find the ratio of the weight of a volume of the body taken so as to include that point to the weight of an equal volume of the standard body; the *limit* of this ratio, when the volume is indefinitely diminished, is the density of the body at the assumed point.

109. It was shewn in Arts. 66 and 68 that there is a point in every body such that, if the particles of the body be acted on by parallel forces in the same direction and this point be fixed, the body will rest in whatever position it be placed.

Now the weight of a body may be considered as the resultant of the weights of the different elementary portions of the body, acting in parallel and vertical lines. In this case the point

above described as the centre of parallel forces is called the *centre of gravity of the body.* We may define the centre of gravity of any system of heavy particles as a point such that if it be supported and the particles rigidly connected with it, the system will rest in any position.

The following are obvious consequences of this proposition.

When a body is placed on a horizontal plane it will not be in equilibrium unless the vertical through the centre of gravity of the body falls within a polygon formed by so joining the points of contact of the body and the plane as to include them all and have no re-entering angle.

When a body is suspended from a point round which it can move freely, it will not rest unless its centre of gravity be in the vertical line passing through the point of suspension.

For the body is acted on by two forces, its own weight which acts vertically through its centre of gravity and the force arising from the fixed point; for equilibrium these forces must act in the same straight line and in opposite directions; thus the centre of gravity must be in the vertical line passing through the point of suspension.

Hence if a body be suspended successively from two points the vertical lines drawn through the points of suspension will both pass through the centre of gravity; therefore the point at which they intersect is the centre of gravity.

If a body be capable of revolving round an axis which is not vertical it will not rest unless the centre of gravity be in the vertical plane passing through the axis. For the body is acted on by its own weight and the forces arising from the fixed axis; by Art. 87, the moment of the weight round the fixed axis must vanish, and this requires the centre of gravity to be in the vertical plane through the fixed axis.

The student will readily perceive as an experimental fact that there is an important difference between the position of equilibrium in which the centre of gravity is vertically *above* the fixed point or fixed axis, and that in which it is vertically *below* it. In the former case, if the body be slightly disturbed from its equilibrium position and then left to itself, it will

begin to recede from its original position. In the latter case, if the body be slightly disturbed from its equilibrium position and then left to itself, it will begin to return to its original position. The former position of equilibrium is called *unstable*, and the latter *stable*. We shall return to this point in Chap. XIV.

We now proceed to determine the position of the centre of gravity in various cases.

(1) *Given the centres of gravity of two parts which compose a body, to find the centre of gravity of the whole body.*

Let G_1 denote the centre of gravity of one part, and G_2 the centre of gravity of the other part; let m_1 denote the mass of the first part and m_2 the mass of the second part. Join G_1G_2 and divide it at G so that $\frac{GG_1}{GG_2} = \frac{m_2}{m_1}$, then G is the centre of gravity of the whole body (Art. 37).

(2) *Given the centre of gravity of a body and also the centre of gravity of a part of the body, to find the centre of gravity of the remainder.*

Let G denote the centre of gravity of the body, and G_1 the centre of gravity of a part of the body; let m denote the mass of the body, and m_1 the mass of the part. Join G_1G and produce it through G to G_2, so that $\frac{GG_2}{GG_1} = \frac{m_1}{m - m_1}$, then G_2 is the centre of gravity of the remainder.

(3) *To find the centre of gravity of a triangular figure of uniform thickness and density.*

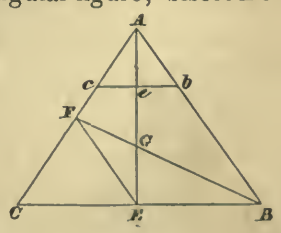
Let ABC be one surface of the triangular figure; bisect BC at E; join AE; draw *ceb* parallel to CEB cutting AE at e. Then, by similar triangles,

$$ce : CE :: Ae : AE,$$

and $$be : BE :: Ae : AE,$$

therefore $$ce : CE :: be : BE;$$

but $CE = BE$, therefore $ce = be$.



Hence AE bisects every straight line parallel to BC. Therefore each of the strips similar to *ceb*, into which we may suppose the triangle to be divided, will balance on AE, and therefore the centre of gravity must be in the straight line AE.

Bisect AC at F and join BF; let this cut AE at G. Then, as before, the centre of gravity must be in BF; but it must be in AE; and therefore G is the centre of gravity.

Join EF. Then, because $CE = BE$ and $CF = AF$, therefore EF is parallel to AB and $AB = 2FE$; and by similar triangles,

$$EG : EF :: AG : AB, \text{ therefore } EG = \tfrac{1}{2}AG.$$

Hence to find the centre of gravity of a triangle, bisect any side, join the point of bisection with the opposite angle, and the centre of gravity lies a third of the way up this straight line.

The centre of gravity of any plane polygon may be found by dividing it into triangles, determining the centre of gravity of each triangle, and then by Art. 66 deducing the centre of gravity of the whole figure.

We may observe that the centre of gravity of a triangle coincides with the centre of gravity of three *equal* particles placed at the angular points of the triangle. For to find the centre of gravity of three equal particles placed at A, B, C respectively, we join CB and bisect it at E; then E is the centre of gravity of the particles at C and B; suppose these particles collected at E; then join AE and divide AE at G so that EG may be to AG as the mass of the one particle at A is to that of the two at E, that is, as 1 is to 2; then G is the centre of gravity of the three equal particles. From the construction G is obviously also the centre of gravity of the triangle ABC.

Let the co-ordinates of A referred to any axes be x_1, y_1, z_1; those of B, x_2, y_2, z_2; and those of C, x_3, y_3, z_3; then, by Art. 66, the co-ordinates $\bar{x}$, $\bar{y}$, $\bar{z}$ of the centre of gravity of three equal particles placed at A, B, C respectively, are

$$\bar{x} = \tfrac{1}{3}(x_1 + x_2 + x_3); \quad \bar{y} = \tfrac{1}{3}(y_1 + y_2 + y_3); \quad \bar{z} = \tfrac{1}{3}(z_1 + z_2 + z_3).$$

By what we have just proved, these are also the co-ordinates of the centre of gravity of the *triangle ABC*.

It may be remarked that in Art. 66 the co-ordinates may be *rectangular or oblique.*

(4) The centre of gravity of any rectilinear figure might be found by cutting it up into triangles, and replacing each triangle by three equal particles at its angular points. If the triangles are all equal the particles will be all equal; but if the triangles are not all equal, the particles which are substituted for any triangle must be taken proportional to the area of that triangle. For example, we will determine the position of the centre of gravity of any quadrilateral.

Let $ABCD$ denote the quadrilateral; let the diagonals AC and BD intersect at E.

Let $BE = n_1$ and $ED = n_2$; then the areas of the triangles ABC and ACD will be proportional to n_1 and n_2, and may be denoted by μn_1 and μn_2 respectively. Let y_1, y_2, y_3, y_4 be the ordinates of A, B, C, D respectively; then replace the triangle ABC by three particles at A, B, C respectively each equal to μn_1; and replace the triangle ACD by three particles at A, D, C respectively each equal to μn_2. Let y denote the ordinate of the centre of gravity of the system, that is, of the given quadrilateral; then

$$y = \frac{\mu n_1 (y_1 + y_2 + y_3) + \mu n_2 (y_3 + y_4 + y_1)}{3 (\mu n_1 + \mu n_2)}$$

$$= \tfrac{1}{3} (y_1 + y_3) + \frac{n_1 y_2 + n_2 y_4}{3 (n_1 + n_2)}.$$

This may be simplified by making use of the ordinate of the point E; denote this by y: then, by *Plane Co-ordinate Geometry*, Art. 10, we have

$$y = \frac{n_2 y_2 + n_1 y_4}{n_1 + n_2};$$

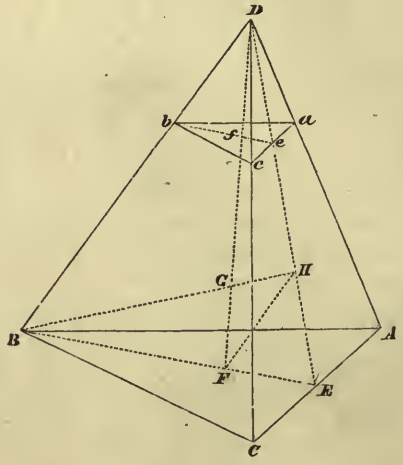
therefore
$$\overline{y} + \frac{y}{3} = \tfrac{1}{3} (y_1 + y_2 + y_3 + y_4),$$

so that
$$\overline{y} = \tfrac{1}{3} (y_1 + y_2 + y_3 + y_4 - y).$$

Similarly the abscissa of the centre of gravity can be expressed in terms of those of A, B, C, D and E.

(5) *To find the centre of gravity of a pyramid on a triangular base.*

Let ABC be the base, D the vertex; bisect AC at E; join BE, DE; take $EF = \frac{1}{3} EB$, then F is the centre of gravity of ABC. Join FD; draw ab, bc, ca parallel to AB, BC, CA



respectively, and let DF meet the plane abc at f; join bf and produce it to meet DE at e. Then, by similar triangles, $ae = ec$; also

$$\frac{bf}{BF} = \frac{Df}{DF} = \frac{ef}{EF};$$

but $EF = \frac{1}{2} BF$, therefore $ef = \frac{1}{2} bf$;

therefore f is the centre of gravity of the triangle abc; and if we suppose the pyramid to be made up of an indefinitely great number of indefinitely thin triangular slices parallel to the base, each of these slices has its centre of gravity in DF. Hence the centre of gravity of the pyramid is in DF.

Again, take $EH = \frac{1}{3} ED$; join HB cutting DF at G. Then, as before, the centre of gravity of the pyramid must be in BH; but it is in DF; hence G, the point of intersection of these straight lines, is the centre of gravity.

Join FH; then FH is parallel to DB. Also because $EF = \frac{1}{3} EB$, therefore $FH = \frac{1}{3} DB$, and

$\frac{FG}{FH} = \frac{DG}{DB}$; but $FH = \frac{1}{3} DB$, therefore $FG = \frac{1}{3} DG = \frac{1}{4} DF$.

Hence the centre of gravity is one-fourth of the way up the straight line joining the centre of gravity of the base with the vertex.

In the same way as the corresponding results were demonstrated for the triangle, we may establish the following:

The centre of gravity of a pyramid coincides with the centre of gravity of particles of equal mass placed at the angular points of the pyramid.

Let x_1, y_1, z_1 be the co-ordinates of one angular point; x_2, y_2, z_2 the co-ordinates of another; and so on; let $\bar{x}, \bar{y}, \bar{z}$ be the co-ordinates of the centre of gravity of the pyramid: then, by Art. 66,

$$\bar{x} = \tfrac{1}{4}(x_1 + x_2 + x_3 + x_4),$$
$$\bar{y} = \tfrac{1}{4}(y_1 + y_2 + y_3 + y_4),$$
$$\bar{z} = \tfrac{1}{4}(z_1 + z_2 + z_3 + z_4).$$

(6) *To find the centre of gravity of any pyramid having a plane base.*

Divide the base into triangles; if any part of the base is curvilinear then suppose the curve to be divided into an indefinitely great number of indefinitely short straight lines. Join the vertex of the pyramid with the centres of gravity of all the triangles, and also with all their angles. Draw a plane parallel to the base at a distance from the base equal to one-fourth of the distance of the vertex from the base; then this plane cuts every straight line drawn from the vertex to the base in parts having the same ratio of 3 to 1; and therefore the triangular pyramids have their centres of gravity in this plane, and therefore the whole pyramid has its centre of gravity in this plane.

Again, join the vertex with the centre of gravity of the base; then every section parallel to the base will be similar to the base, and if we suppose the pyramid divided into an indefinitely large number of indefinitely thin slices by planes

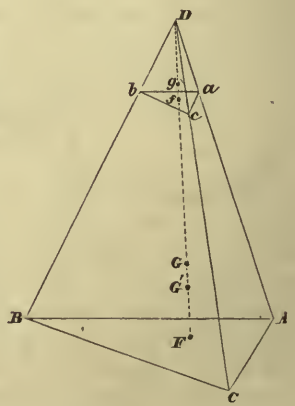
parallel to the base, the centre of gravity of each slice will lie on the straight line joining the vertex with the centre of gravity of the base. Hence the whole pyramid has its centre of gravity in this straight line.

Therefore the centre of gravity is one-fourth of the way up the straight line joining the centre of gravity of the base with the vertex.

This result will hold also for any cone, right or oblique; for the cone may be decomposed into an infinite number of infinitesimal pyramids having a common vertex.

(7) *To find the centre of gravity of the frustum of a pyramid formed by parallel planes.*

Let $ABCabc$ be the frustum; G, g the centres of gravity of the pyramids $DABC$, $Dabc$; it is clear that the centre of gravity of the frustum must be in gG produced; suppose it at G'.



Let $Ff = c$, $AB = a$, $ab = b$.

Since the whole pyramid $DABC$ is made up of the frustum and the small pyramid, therefore,

$$\frac{GG'}{Gg} = \frac{\text{weight of small pyramid}}{\text{weight of frustum}}$$

$$= \frac{\text{vol. of small pyr.}}{\text{vol. of large pyr.} - \text{vol. of small pyr.}} = \frac{b^3}{a^3 - b^3},$$

since similar solids are as the cubes of their homologous edges;

and $$Gg = DG - Dg = \tfrac{3}{4}(DF - Df) = \tfrac{3}{4}c;$$

$$\text{therefore } GG' = \frac{3c}{4} \cdot \frac{b^3}{a^3 - b^3}.$$

Also $GF = \frac{1}{4}DF = \frac{1}{4}(DF - Df)\frac{a}{a-b}$ by similar figures,

$$= \frac{c}{4} \cdot \frac{a}{a-b};$$

therefore $FG' = FG - G'G = \frac{c}{4}\left\{\frac{a}{a-b} - \frac{3b^3}{a^3-b^3}\right\}$

$$= \frac{c}{4}\,\frac{a^2+2ab+3b^2}{a^2+ab+b^2}.$$

This is true of a frustum of a pyramid on any base, a and b being homologous sides of the two ends.

(8) We may by the aid of the theory of the centre of gravity demonstrate some geometrical propositions. For example: *the straight lines which join the middle points of the opposite edges of a tetrahedron meet at a point which bisects each straight line.*

For suppose equal particles placed at the corners of a tetrahedron; then to find the centre of gravity of the system we may proceed thus: The centre of gravity of any pair of particles is at the middle point of the edge which joins them; and the centre of gravity of the other pair is at the middle point of the opposite edge: then the centre of gravity of the system is at the middle point of the straight line which joins the middle points of the selected edges. And the same point will of course be obtained for the centre of gravity of the system, whatever pair of edges be selected. Hence the required result is obtained.

(9) *Particles are placed at the corners of a tetrahedron, the mass of each particle being proportional to the area of the opposite face: shew that the centre of gravity of the system coincides with the centre of the sphere inscribed in the tetrahedron.*

Let A, B, C, D be the angular points of the tetrahedron. Let p be the perpendicular from D on the face ABC. Then the distance of the centre of gravity of the system from the plane ABC

$$= \frac{p \times \text{area of face } ABC}{\text{sum of the areas of the faces}}$$

$$= \frac{3 \times \text{volume of tetrahedron}}{\text{sum of the areas of the faces}}.$$

And this expression is equal to the radius of the sphere inscribed in the tetrahedron.

Hence the required result is obtained.

(10) *A polyhedron is circumscribed about a sphere; at the points of contact masses are placed which are proportional to the areas of the corresponding faces of the polyhedron: shew that the centre of gravity of these masses coincides with the centre of the sphere.*

Take the centre of the sphere for origin, and any plane through the origin for the plane of (x, y).

Let $A_1, A_2, A_3, \ldots\ldots$ denote the areas of the faces of the polyhedron; let $z_1, z_2, z_3, \ldots$ denote the ordinates of the points of contact; $\bar{z}$ the ordinate of the centre of gravity. Then, by Art. 66,

$$\bar{z} = \frac{A_1 z_1 + A_2 z_2 + A_3 z_3 + \ldots}{A_1 + A_2 + A_3 + \ldots}.$$

Now the projection of the area A_1 on the plane of (x, y) is $\frac{A_1 z_1}{r}$, where r is the radius of the sphere; and similarly for the other projections. And the sum of such projections is zero. Thus $\bar{z} = 0$; and since the plane of (x, y) is any plane through the centre of the sphere, the centre of gravity must coincide with the centre of the sphere.

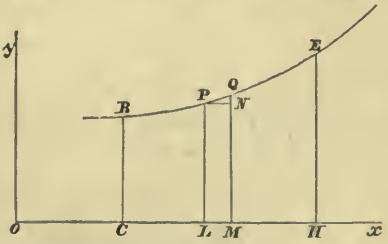
We proceed now to the analytical calculations.

110. In all the cases in which the Integral Calculus is employed to ascertain the centre of gravity of a body the principle is the same; the body is divided into an indefinitely large number of indefinitely small elements; the volume of an element is estimated, and this being multiplied by the

density gives the mass of the element. The mass is multiplied by the abscissa of the element, and we find the sum of the values of this product for all the elements; the result corresponds to the ΣPx of Art. 66. Also we find the sum of the masses of all the elements and thus obtain a result corresponding to the ΣP of the same Article. Divide the former result by the latter and we have the value of $\bar{x}$; similarly $\bar{y}$ and $\bar{z}$ can be found. In the following examples the student must not allow the details of the Integral Calculus to obscure his recognition of the fundamental formula of Art. 66; he must consider in every case what corresponds to the P, x, y, z of that Article, that is, he must carefully ascertain into what elements the body is decomposed.

Plane Area.

111. Let $CBEH$ be an area bounded by the ordinates BC and EH, the curve BE, and the portion CH of the axis of x; it is required to find the centre of gravity of the area. Or instead of the area we may ask for the centre of gravity of a solid bounded by two planes parallel to the plane of the paper and equidistant from it, and by a straight line which moves round the boundary $CBEH$ remaining always perpendicular to the plane of the paper. Divide CH into n portions, and suppose ordinates drawn at the points of division. Let LP and MQ represent two consecutive ordinates, and draw PN parallel to LM.



Let $OL = x$, $LP = y$, $LM = \Delta x$, $OC = c$, $OH = h$.

The area of the rectangle PM is $y\Delta x$; suppose u to denote the area of PQN, and let x' be the abscissa of the centre of gravity of the area $PQLM$. Then if k denote the thickness of the solid and ρ its density, $k\rho(y\Delta x + u)$ is the mass of the

element $PQML$. Hence, if $\bar{x}$ be the abscissa of the centre of gravity of the whole figure $CBEH$, by Art. 66,

$$\bar{x} = \frac{\Sigma k\rho x'(y\Delta x + u)}{\Sigma k\rho (y\Delta x + u)} = \frac{\Sigma x'(y\Delta x + u)}{\Sigma (y\Delta x + u)},$$

supposing the thickness and density uniform. The summation is to include all the figures like $PQML$, which are comprised in $CBEH$.

Now suppose n to increase without limit, and each of the portions LM to diminish without limit; then the term Σu in the denominator of $\bar{x}$ vanishes; for it expresses the sum of all the figures like PQN, and is therefore less than a rectangle having for its breadth Δx and for its height the difference of the extreme ordinates CB and HE. Also the term $\Sigma x'u$ in the numerator of $\bar{x}$ vanishes, for it is less than the product $h\Sigma u$, and as we have just shewn, this ultimately vanishes. Hence the expression for $\bar{x}$ becomes, when the number of divisions is indefinitely increased and each term indefinitely diminished,

$$\frac{\Sigma x' y\Delta x}{\Sigma y\Delta x}.$$

Moreover, x' must lie between x and $x+\Delta x$: suppose it equal to $x+v$, where v is less than Δx; then the numerator of $\bar{x}$ may be written

$$\Sigma xy\Delta x + \Sigma vy\Delta x;$$

and as the latter term cannot be so great as $\Delta x\Sigma y\Delta x$, it ultimately vanishes. Hence we have

$$\bar{x} = \frac{\Sigma xy\Delta x}{\Sigma y\Delta x};$$

that is, the above formula will give the correct value of $\bar{x}$ *when we increase the number of divisions indefinitely and diminish each term indefinitely, and extend the summation over the space CBEH.* This will be expressed according to the ordinary notation of the Integral Calculus thus,

$$\bar{x} = \frac{\int_c^h xydx}{\int_c^h ydx} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

In the same manner we may shew that

$$\bar{y} = \frac{\int_c^h y' y dx}{\int_c^h y dx},$$

where y' is the limiting value of the ordinate of the centre of gravity of the element $PQML$ when its breadth is indefinitely diminished; y' is therefore $= \frac{1}{2}y$; hence

$$\bar{y} = \frac{\frac{1}{2}\int_c^h y^2 dx}{\int_c^h y dx} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

We have now only to substitute in (1) and (2) for y its value in terms of x, and then to effect the integration by the ordinary methods.

112. It will not be necessary for the student in solving an example to repeat the whole of the preceding process. When he understands how the proof can be made rigorous, if required, he may proceed shortly thus. The figure $PQML = y\Delta x$ ultimately, and the co-ordinates of its centre of gravity are x and $\frac{1}{2}y$ ultimately. Hence

$$\bar{x} = \frac{\int xy dx}{\int y dx} \text{ and } \bar{y} = \frac{\int \frac{1}{2} yy dx}{\int y dx},$$

the integrations being taken between proper limits.

Unless the contrary be specified, we shall hereafter suppose the bodies we consider to be of *uniform density*, and shall therefore not introduce any factor to represent the density, because, as in the preceding Article, the factor will disappear.

113. Ex. 1. Let the curve be a parabola whose equation is

$$y = 2\sqrt{(ax)}.$$

Here $\bar{x} = \frac{\int_c^h yx dx}{\int_c^h y dx} = \frac{\int_c^h 2\sqrt{(ax)}\, x dx}{\int_c^h 2\sqrt{(ax)}\, dx} = \frac{\int_c^h x^{\frac{3}{2}} dx}{\int_c^h x^{\frac{1}{2}} dx}$

$$= \frac{\frac{2}{5}(h^{\frac{5}{2}} - c^{\frac{5}{2}})}{\frac{2}{3}(h^{\frac{3}{2}} - c^{\frac{3}{2}})}.$$

If $c=0$, $\bar{x}=\frac{3}{5}h$, which determines the abscissa of the centre of gravity of a portion of a parabolic area beginning at the vertex. Also

$$\bar{y}=\frac{\frac{1}{2}\int_c^h y^2dx}{\int_c^h ydx}=\frac{2a\int_c^h xdx}{2\sqrt{a}\int_c^h x^{\frac{1}{2}}dx}=\frac{\sqrt{a}\int_c^h xdx}{\int_c^h x^{\frac{1}{2}}dx}=\frac{\frac{1}{2}\sqrt{a}\,(h^2-c^2)}{\frac{2}{3}(h^{\frac{3}{2}}-c^{\frac{3}{2}})}.$$

When $c=0$, $\bar{y}=\frac{3}{4}\sqrt{(ah)}$.

Ex. 2. Let the curve be an ellipse whose equation is

$$y=\frac{b}{a}\sqrt{(a^2-x^2)}.$$

Here $$\bar{x}=\frac{\int_c^h yxdx}{\int_c^h ydx}=\frac{\int_c^h \frac{bx}{a}\sqrt{(a^2-x^2)}\,dx}{\int_c^h \frac{b}{a}\sqrt{(a^2-x^2)}\,dx}=\frac{\int_c^h x\sqrt{(a^2-x^2)}\,dx}{\int_c^h \sqrt{(a^2-x^2)}\,dx}.$$

Now $$\int x\sqrt{(a^2-x^2)}\,dx=-\tfrac{1}{3}(a^2-x^2)^{\frac{3}{2}};$$

therefore $\int_c^h x\sqrt{(a^2-x^2)}\,dx=\frac{1}{3}(a^2-c^2)^{\frac{3}{2}}-\frac{1}{3}(a^2-h^2)^{\frac{3}{2}}$.

And $$\int\sqrt{(a^2-x^2)}\,dx=\frac{x\sqrt{(a^2-x^2)}}{2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a};$$

therefore

$$\int_c^h\sqrt{(a^2-x^2)}\,dx=\frac{h\sqrt{(a^2-h^2)}-c\sqrt{(a^2-c^2)}}{2}+\frac{a^2}{2}\left(\sin^{-1}\frac{h}{a}-\sin^{-1}\frac{c}{a}\right).$$

Hence $\bar{x}$ is known.

Also $$\bar{y}=\frac{\int_c^h\frac{y^2}{2}dx}{\int_c^h ydx}=\frac{\frac{b}{2a}\int_c^h(a^2-x^2)\,dx}{\int_c^h\sqrt{(a^2-x^2)}\,dx}$$

$$=\frac{\frac{b}{2a}\left\{a^2(h-c)-\frac{h^3-c^3}{3}\right\}}{\frac{h\sqrt{(a^2-h^2)}-c\sqrt{(a^2-c^2)}}{2}+\frac{a^2}{2}\left(\sin^{-1}\frac{h}{a}-\sin^{-1}\frac{c}{a}\right)}.$$

If we require the centre of gravity of the quadrant of the ellipse, we must put $c = 0$ and $h = a$. Hence

$$\bar{x} = \frac{4a}{3\pi}, \quad \bar{y} = \frac{4b}{3\pi}.$$

Ex. 3. Let the curve be a cycloid whose equation is

$$y = \sqrt{(2ax - x^2)} + a \operatorname{vers}^{-1} \frac{x}{a};$$

and suppose we require the centre of gravity of half the area of the curve; then

$$\bar{x} = \frac{\int_0^{2a} yx\,dx}{\int_0^{2a} y\,dx}, \quad \bar{y} = \frac{\frac{1}{2}\int_0^{2a} y^2\,dx}{\int_0^{2a} y\,dx}.$$

Now
$$\int yx\,dx = \frac{yx^2}{2} - \int \frac{x^2}{2}\frac{dy}{dx}\,dx$$
$$= \frac{yx^2}{2} - \int \frac{x^2}{2}\sqrt{\left(\frac{2a-x}{x}\right)}\,dx.$$

Also, when $x = 0$, $y = 0$, and when $x = 2a$, $y = \pi a$;

therefore $\int_0^{2a} yx\,dx = \frac{1}{2}\{\pi a\,(2a)^2\} - \frac{1}{2}\int_0^{2a} x\sqrt{\,}(2ax - x^2)\,dx$:

and as $\int_0^{2a} x\sqrt{(2ax - x^2)}\,dx$ will be found $= \frac{1}{2}\pi a^3$, we have

$$\int_0^{2a} yx\,dx = 2\pi a^3 - \tfrac{1}{4}\pi a^3 = \tfrac{7}{4}\pi a^3.$$

Again,
$$\int y\,dx = yx - \int x\frac{dy}{dx}\,dx$$
$$= yx - \int \sqrt{(2ax - x^2)}\,dx;$$

therefore
$$\int_0^{2a} y\,dx = 2\pi a^2 - \int_0^{2a} \sqrt{(2ax - x^2)}\,dx$$
$$= 2\pi a^2 - \tfrac{1}{2}\pi a^2 = \tfrac{3}{2}\pi a^2;$$

therefore
$$\bar{x} = \frac{\frac{7}{4}\pi a^3}{\frac{3}{2}\pi a^2} = \tfrac{7}{6}a.$$

Also

$$\int y^2 dx = y^2 x - 2\int yx \frac{dy}{dx} dx$$

$$= y^2 x - 2\int y \sqrt{(2ax - x^2)}\, dx$$

$$= y^2 x - 2\int (2ax - x^2)\, dx - 2a\int \sqrt{(2ax - x^2)}\, \text{vers}^{-1}\frac{x}{a} dx$$

$$= y^2 x - 2ax^2 + \frac{2x^3}{3} - 2a\int \sqrt{(2ax - x^2)}\, \text{vers}^{-1}\frac{x}{a} dx\,;$$

therefore $\int_0^{2a} y^2 dx = 2\pi^2 a^3 - \frac{8a^3}{3} - 2a\int_0^{2a} \sqrt{(2ax - x^2)}\, \text{vers}^{-1}\frac{x}{a} dx.$

By assuming $\text{vers}^{-1}\frac{x}{a} = \theta$, we may shew that

$$\int_0^{2a} \sqrt{(2ax - x^2)}\, \text{vers}^{-1}\frac{x}{a} dx = \frac{\pi^2 a^2}{4}.$$

Hence $$\int_0^{2a} y^2 dx = \tfrac{3}{2}\pi^2 a^3 - \tfrac{8}{3}a^3\,;$$

therefore $$\bar{y} = \frac{\frac{1}{2}a\,(\frac{3}{2}\pi^2 - \frac{8}{3})}{\frac{3}{2}\pi} = \frac{a}{3\pi}(\tfrac{3}{2}\pi^2 - \tfrac{8}{3}).$$

114. If a curve have a branch below the axis of x symmetrical with one above the axis, and we require the centre of gravity of the area bounded by the two branches and ordinates drawn at the distances c and h from the origin, we have

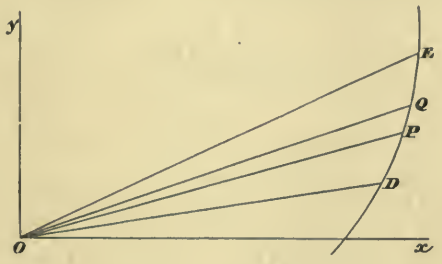
$$\bar{x} = \frac{2\int_c^h yx dx}{\int_c^h y dx} = \frac{\int_c^h yx dx}{\int_c^h y dx},$$

and $$\bar{y} = 0.$$

115. We have hitherto supposed the axes *rectangular*; if they are oblique and inclined at an angle ω, then the figure $PQML$ (see fig. to Art. 111) will $= \sin\omega\, y\Delta x$ ultimately. Hence the formulæ (1) and (2) of Art. 111 remain true, for $\sin\omega$ occurs as a factor in the numerator and denominator, and may therefore be cancelled.

116. It is sometimes convenient to use polar formulæ.

Let DE be the arc of a curve; and suppose we require the centre of gravity of the area comprised between the arc DE and the radii OD, OE drawn from the pole O.



Divide the angle DOE into a number of angles, of which POQ represents one; let $OP = r$, $POx = \theta$, $POQ = \Delta\theta$. The area $POQ = \frac{1}{2}r^2\Delta\theta$ ultimately (*Differential Calculus*, Art. 313). Also the centre of gravity of the figure POQ will be ultimately, like that of a triangle, on a straight line drawn from O bisecting the chord PQ, and at a distance of two-thirds of this straight line from O. Hence the abscissa and ordinate of the centre of gravity of POQ will be ultimately

$$\tfrac{2}{3}r\cos\theta, \text{ and } \tfrac{2}{3}r\sin\theta \text{ respectively.}$$

Hence
$$\bar{x} = \frac{\int \frac{2}{3}r\cos\theta\,\frac{1}{2}r^2 d\theta}{\int \frac{1}{2}r^2 d\theta} = \frac{\frac{2}{3}\int r^3\cos\theta d\theta}{\int r^2 d\theta},$$

$$\bar{y} = \frac{\int \frac{2}{3}r\sin\theta\,\frac{1}{2}r^2 d\theta}{\int \frac{1}{2}r^2 d\theta} = \frac{\frac{2}{3}\int r^3\sin\theta d\theta}{\int r^2 d\theta}.$$

In these formulæ we must put for r its value in terms of θ given by the equation to the curve; we must then integrate from $\theta = \alpha$ to $\theta = \beta$, supposing α and β the angles which OD and OE respectively make with the fixed straight line Ox.

117. Ex. Let O be the focus of a parabola, and the fixed straight line Ox pass through the vertex ; then

$$r = \frac{a}{\cos^2 \frac{1}{2}\theta},$$

where $4a$ is the *latus rectum* of the parabola.

Hence
$$\bar{x} = \frac{\frac{2}{3}a \int_\alpha^\beta \frac{\cos\theta}{\cos^6 \frac{1}{2}\theta}\, d\theta}{\int_\alpha^\beta \frac{d\theta}{\cos^4 \frac{1}{2}\theta}}.$$

Now $$\int \frac{\cos\theta}{\cos^6 \frac{1}{2}\theta}\, d\theta = \int \frac{\cos^2 \frac{1}{2}\theta - \sin^2 \frac{1}{2}\theta}{\cos^4 \frac{1}{2}\theta} \sec^2 \tfrac{1}{2}\theta d\theta$$

$$= \int (1 - \tan^2 \tfrac{1}{2}\theta)(1 + \tan^2 \tfrac{1}{2}\theta) \sec^2 \tfrac{1}{2}\theta d\theta$$

$$= \int (1 - \tan^4 \tfrac{1}{2}\theta) \sec^2 \tfrac{1}{2}\theta d\theta = 2(\tan \tfrac{1}{2}\theta - \tfrac{1}{5}\tan^5 \tfrac{1}{2}\theta);$$

therefore $$\int_\alpha^\beta \frac{\cos\theta}{\cos^6 \frac{1}{2}\theta} d\theta = 2(\tan \tfrac{1}{2}\beta - \tan \tfrac{1}{2}\alpha) - \tfrac{2}{5}(\tan^5 \tfrac{1}{2}\beta - \tan^5 \tfrac{1}{2}\alpha).$$

Also $$\int \frac{d\theta}{\cos^4 \frac{1}{2}\theta} = \int (1 + \tan^2 \tfrac{1}{2}\theta) \sec^2 \tfrac{1}{2}\theta d\theta = 2\tan \tfrac{1}{2}\theta + \tfrac{2}{3}\tan^3 \tfrac{1}{2}\theta;$$

therefore $$\int_\alpha^\beta \frac{d\theta}{\cos^4 \frac{1}{2}\theta} = 2(\tan \tfrac{1}{2}\beta - \tan \tfrac{1}{2}\alpha) + \tfrac{2}{3}(\tan^3 \tfrac{1}{2}\beta - \tan^3 \tfrac{1}{2}\alpha);$$

therefore $$\bar{x} = \tfrac{2}{3}a \cdot \frac{\tan \frac{1}{2}\beta - \tan \frac{1}{2}\alpha - \frac{1}{5}(\tan^5 \frac{1}{2}\beta - \tan^5 \frac{1}{2}\alpha)}{\tan \frac{1}{2}\beta - \tan \frac{1}{2}\alpha + \frac{1}{3}(\tan^3 \frac{1}{2}\beta - \tan^3 \frac{1}{2}\alpha)}.$$

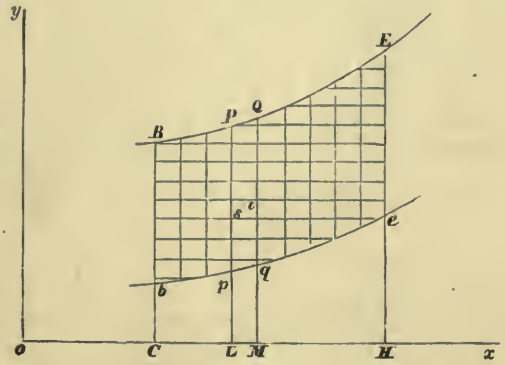
Again, $$\int \frac{\sin\theta}{\cos^6 \frac{1}{2}\theta} d\theta = 2\int \frac{\sin \frac{1}{2}\theta}{\cos^5 \frac{1}{2}\theta} d\theta = \frac{1}{\cos^4 \frac{1}{2}\theta};$$

therefore $$\int_\alpha^\beta \frac{\sin\theta}{\cos^6 \frac{1}{2}\theta} d\theta = \sec^4 \tfrac{1}{2}\beta - \sec^4 \tfrac{1}{2}\alpha;$$

therefore $$\bar{y} = \tfrac{1}{3}a \cdot \frac{\sec^4 \frac{1}{2}\beta - \sec^4 \frac{1}{2}\alpha}{\tan \frac{1}{2}\beta - \tan \frac{1}{2}\alpha + \frac{1}{3}(\tan^3 \frac{1}{2}\beta - \tan^3 \frac{1}{2}\alpha)}.$$

Plane Area. Double Integration.

118. There is another method of dividing a plane area into elements, to which we now proceed.



Let a series of straight lines be drawn parallel to the axis of y, and another series of straight lines parallel to the axis of x. Let st represent one of the rectangles formed by these straight lines; and suppose x and y to be the co-ordinates of s, and $x + \Delta x$ and $y + \Delta y$ the co-ordinates of t. Then the area of the rectangle st is $\Delta x \Delta y$, and the co-ordinates of its centre of gravity are ultimately x and y. Hence, to find the abscissa of the centre of gravity of any plane area, we can take the sum of the values of $x\Delta x\Delta y$ for the numerator, and the sum of the values of $\Delta x \Delta y$ for the denominator, Δx and Δy being indefinitely diminished. This is expressed thus,

$$\overline{x} = \frac{\iint x dx dy}{\iint dx dy}.$$

Similarly,

$$\overline{y} = \frac{\iint y dx dy}{\iint dx dy}.$$

119. Suppose, for example, that the area is bounded by the two ordinates *BbC*, *EeH*, and the two curves *BPQE*, *bpqe*. Let $y = \phi(x)$ be the equation to the upper curve, and $y = \psi(x)$ the equation to the lower curve; let $OC = c$, $OH = h$. The sum of the product $x\Delta x\Delta y$ for all the rectangles similar to *st*, which are contained in the strip *PQqp*, is equal to $x\Delta x$ multiplied by the sum of the values of Δy, for $x\Delta x$ has the same value for each of these rectangles. Since the sum of the values of Δy is *Pp* or $\phi(x) - \psi(x)$, we have $x\Delta x.\{\phi(x) - \psi(x)\}$ as the result obtained by considering all the rectangles in the strip *PQqp*. We have then to sum up the values of $x\Delta x\{\phi(x) - \psi(x)\}$ for all the strips similar to *PQqp* comprised between *Bb* and *Ee*; that is, we must determine the value of $\int_c^h x\{\phi(x) - \psi(x)\}\,dx$. Considerations of a similar kind apply to the denominator of $\bar{x}$, and we obtain

$$\bar{x} = \frac{\int_c^h x\{\phi(x) - \psi(x)\}\,dx}{\int_c^h \{\phi(x) - \psi(x)\}\,dx}.$$

In the numerator of $\bar{y}$ we observe that $y\Delta y\Delta x$ represents that portion of it which arises from the element *st*; hence we shall find the result obtained from all the elements in the strip *PQqp*, if we determine the sum of all the values of $y\Delta y$, and multiply the result by Δx. Now the sum of the values of $y\Delta y$ is $\int_{\psi(x)}^{\phi(x)} y\,dy$, or $\frac{1}{2}[\{\phi(x)\}^2 - \{\psi(x)\}^2]$. If we multiply by Δx, and find the sum of the values of the product for all the strips between *Bb* and *Ee*, we obtain the numerator of $\bar{y}$. Hence

$$y = \frac{\frac{1}{2}\int_c^h [\{\phi(x)\}^2 - \{\psi(x)\}^2]\,dx}{\int_c^h \{\phi(x) - \psi(x)\}\,dx}.$$

The value of $\bar{y}$ may be written thus

$$\bar{y} = \frac{\int_c^h \frac{1}{2}\{\phi(x) + \psi(x)\}\{\phi(x) - \psi(x)\}\,dx}{\int_c^h \{\phi(x) - \psi(x)\}\,dx}.$$

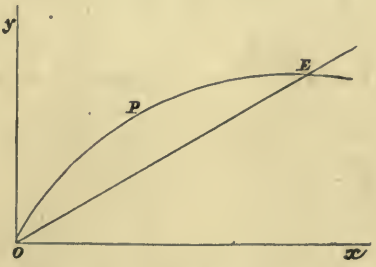
The meaning of the factors in the numerator is now apparent; for $\{\phi(x) - \psi(x)\}\,\Delta x$ ultimately represents the area of the strip *PQqp*, and $\frac{1}{2}\{\phi(x) + \psi(x)\}$, which is the ordinate of the middle point of *Pp*, will ultimately be the ordinate

of the centre of gravity of $PQqp$. Hence the above equation agrees with that given in Art. 66,

$$\bar{y} = \frac{\Sigma Py}{\Sigma P}.$$

The process and the figure in the preceding two Articles would have been unnecessary if our only object had been to establish the formulæ for $\bar{x}$ and $\bar{y}$, since these formulæ can be obtained more simply as we have just shewn. But we shall require hereafter other formulæ involving double integration, and have therefore directed the reader's attention to these in order to accustom him to the subject.

120. Ex. Let OPE be a parabola having for its equation $y^2 = 4ax$, and OE a straight line having for its equation $y = kx$; find the centre of gravity of the area OPE between the curve and the straight line.



Here $\phi(x) = 2\sqrt{(ax)}$, $\psi(x) = kx$, $c = 0$; h is to be found from the equation $2\sqrt{(ah)} = kh$;

therefore
$$h = \frac{4a}{k^2}.$$

Thus
$$\bar{x} = \frac{\int_0^h x\{2\sqrt{(ax)} - kx\}\,dx}{\int_0^h \{2\sqrt{(ax)} - kx\}\,dx}$$

$$= \frac{\frac{4}{5}\sqrt{a}h^{\frac{5}{2}} - \frac{1}{3}kh^3}{\frac{4}{3}\sqrt{a}h^{\frac{3}{2}} - \frac{1}{2}kh^2} = h\frac{\frac{4}{5}\sqrt{a} - \frac{1}{3}k\sqrt{h}}{\frac{4}{3}\sqrt{a} - \frac{1}{2}k\sqrt{h}} = h\frac{\frac{4}{5} - \frac{2}{3}}{\frac{4}{3} - 1}$$

$$= \frac{2h}{5} = \frac{8a}{5k^2}.$$

Similarly, $\bar{y} = \dfrac{\frac{1}{2}\int_0^h (4ax - k^2x^2)\,dx}{\int_0^h \{2\sqrt{(ax)} - kx\}\,dx}$

$$= \frac{\frac{1}{2}(2ah^2 - \frac{1}{3}k^2h^3)}{\frac{4}{3}\sqrt{a}h^{\frac{3}{2}} - \frac{1}{2}kh^2} = \frac{\frac{1}{2}\sqrt{h}\,(2a - \frac{1}{3}k^2h)}{\frac{4}{3}\sqrt{a} - \frac{1}{2}k\sqrt{h}}$$

$$= \frac{\frac{1}{2}\sqrt{h}\,(2a - \frac{4}{3}a)}{\frac{4}{3}\sqrt{a} - \sqrt{a}} = \sqrt{(ah)} = \frac{2a}{k}.$$

121. Sometimes it will be more convenient to integrate the formulæ in Art. 118, *first* with respect to x and then with respect to y. For example, if the given area is comprised between the straight lines $y = c'$, and $y = h'$, and the curves $x = \psi(y)$, and $x = \phi(y)$, we obtain

$$\bar{x} = \frac{\frac{1}{2}\int_{c'}^{h'} [\{\phi(y)\}^2 - \{\psi(y)\}^2]\,dy}{\int_{c'}^{h'} \{\phi(y) - \psi(y)\}\,dy},$$

$$\bar{y} = \frac{\int_{c'}^{h'} y\,\{\phi(y) - \psi(y)\}\,dy}{\int_{c'}^{h'} \{\phi(y) - \psi(y)\}\,dy}.$$

If we apply these to the example given in Art. 120, we have $\psi(y) = \dfrac{y^2}{4a}$, $\phi(y) = \dfrac{y}{k}$, $c' = 0$, and h' is to be found from the equation $\dfrac{h'}{k} = \dfrac{h'^2}{4a}$; therefore $h' = \dfrac{4a}{k}$.

Hence

$$\bar{x} = \frac{\frac{1}{2}\int_0^{h'} \left(\frac{y^2}{k^2} - \frac{y^4}{16a^2}\right) dy}{\int_0^{h'} \left(\frac{y}{k} - \frac{y^2}{4a}\right) dy},$$

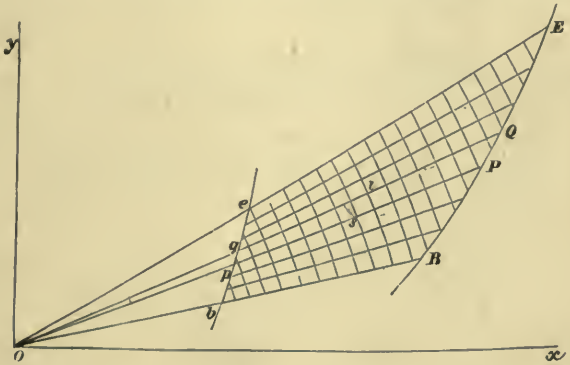
$$\bar{y} = \frac{\int_0^{h'} y \left(\frac{y}{k} - \frac{y^2}{4a}\right) dy}{\int_0^{h'} \left(\frac{y}{k} - \frac{y^2}{4a}\right) dy}.$$

The results will of course be the same as before.

For fuller explanations and illustrations of double integrations the student is referred to treatises on the Integral Calculus. (See especially *Integral Calculus*, Art. 141 and Art. 152.)

122. We will now give polar formulæ involving double integration.

Let a series of straight lines be drawn from a pole O, also a series of circles be described from O as a centre. Let st be one of the elements formed by this mode of dividing a plane area; let r and θ be the polar co-ordinates of s, $r+\Delta r$ and $\theta+\Delta\theta$ the co-ordinates of t; then the area of the element st will be ulti-



mately $r\Delta\theta\,\Delta r$, and the abscissa and ordinate of its centre of gravity will be $r\cos\theta$ and $r\sin\theta$ respectively. Hence we obtain

$$\bar{x} = \frac{\iint r\cos\theta r\,d\theta\,dr}{\iint r\,d\theta\,dr} = \frac{\iint r^2\cos\theta\,d\theta\,dr}{\iint r\,d\theta\,dr}.$$

Similarly $\bar{y} = \dfrac{\iint r^2\sin\theta\,d\theta\,dr}{\iint r\,d\theta\,dr}$.

Suppose the area bounded by the curves $BPQE$, $bpqe$, and the radii ObB, OeE. Let $r=\phi(\theta)$ be the equation to the first curve, $r=\psi(\theta)$ that to the second; and let α and β be the angles which OB and OE make respectively with Ox.

The sum of the values of $r^2\cos\theta\,\Delta r\,\Delta\theta$ for all the elements comprised in the strip $PQqp$, will be found by multiplying the sum of the values of $r^2\Delta r$ by $\cos\theta\Delta\theta$; the former sum is ultimately

$$\int_{\psi(\theta)}^{\phi(\theta)} r^2 dr \text{ or } \tfrac{1}{3}\left[\{\phi(\theta)\}^3 - \{\psi(\theta)\}^3\right].$$

Hence the numerator of the value of $\bar{x}$ is

$$\tfrac{1}{3}\int_\alpha^\beta \cos\theta\,[\{\phi(\theta)\}^3 - \{\psi(\theta)\}^3]\,d\theta,$$

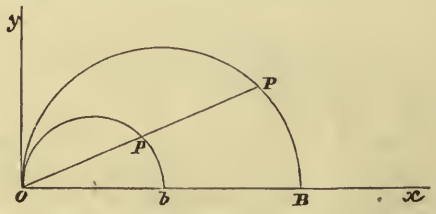
and the denominator, in like manner, is

$$\tfrac{1}{2}\int_\alpha^\beta [\{\phi(\theta)\}^2 - \{\psi(\theta)\}^2]\,d\theta;$$

therefore $$\bar{x} = \frac{\frac{2}{3}\int_\alpha^\beta \cos\theta\,[\{\phi(\theta)\}^3 - \{\psi(\theta)\}^3]\,d\theta}{\int_\alpha^\beta [\{\phi(\theta)\}^2 - \{\psi(\theta)\}^2]\,d\theta}.$$

Similarly, $$\bar{y} = \frac{\frac{2}{3}\int_\alpha^\beta \sin\theta\,[\{\phi(\theta)\}^3 - \{\psi(\theta)\}^3]\,d\theta}{\int_\alpha^\beta [\{\phi(\theta)\}^2 - \{\psi(\theta)\}^2]\,d\theta}.$$

123. Ex. 1. Find the centre of gravity of the area comprised between two semicircles *Opb* and *OPB*.



Let $Ob = c$, $OB = h$; $\phi(\theta) = h\cos\theta$, $\psi(\theta) = c\cos\theta$; $\alpha = 0$, $\beta = \frac{1}{2}\pi$; thus

$$\bar{x} = \frac{\frac{2}{3}(h^3 - c^3)\int_0^{\frac{1}{2}\pi} \cos^4\theta d\theta}{(h^2 - c^2)\int_0^{\frac{1}{2}\pi} \cos^2\theta d\theta}$$

$$= \frac{2}{3}\cdot\frac{3}{4}\cdot\frac{h^3 - c^3}{h^2 - c^2}$$

$$= \frac{1}{2}\frac{h^2 + hc + c^2}{h + c}.$$

(See *Integral Calculus*, Art. 35.)

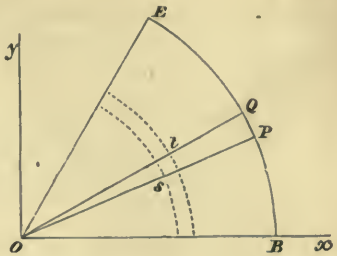
Also $$\bar{y} = \frac{\frac{2}{3}(h^3 - c^3)\int_0^{\frac{1}{2}\pi} \sin\theta\cos^3\theta d\theta}{(h^2 - c^2)\int_0^{\frac{1}{2}\pi} \cos^2\theta d\theta}$$

$$= \frac{\frac{2}{3}(h^3 - c^3)\frac{1}{4}}{(h^2 - c^2)\frac{1}{4}\pi} = \frac{2(h^2 + hc + c^2)}{3(h + c)\pi}.$$

Ex. 2. The sector of a circle,

Let BOE be the sector, subtending an angle β, $OB = a$.

In this example we may with equal facility integrate first with respect to θ and then with respect to r, or first with respect to r and then with respect to θ.



$$\bar{x} = \frac{\int_0^a \int_0^\beta r^2 \cos\theta \, dr \, d\theta}{\int_0^a \int_0^\beta r \, dr \, d\theta} = \frac{\sin\beta \int_0^a r^2 dr}{\beta \int_0^a r \, dr} = \frac{2a \sin\beta}{3\beta},$$

$$\bar{y} = \frac{\int_0^a \int_0^\beta r^2 \sin\theta \, dr \, d\theta}{\int_0^a \int_0^\beta r \, dr \, d\theta} = \frac{(1-\cos\beta) \int_0^a r^2 dr}{\beta \int_0^a r \, dr} = \frac{2a(1-\cos\beta)}{3\beta}.$$

It will be instructive for the student also to notice the solution of this example when rectangular formulæ are used. The equation to the straight line OE is $y = x \tan\beta$; and the equation to the circle EB is $x^2 + y^2 = a^2$.

If we integrate with respect to x first we must integrate from $x = y\cot\beta$ to $x = \sqrt{(a^2 - y^2)}$; since when we integrate with respect to x we have to collect all the elements in a strip which is parallel to the axis of x, and is bounded by OE at one end and by EB at the other. These strips extend from the axis of x up to E, and the ordinate of E is $a \sin\beta$. Hence we integrate with respect to y from $y = 0$ to $y = a \sin\beta$. Therefore

$$\bar{x} = \frac{\int_0^{h'} \int_{\psi(y)}^{\phi(y)} x \, dy \, dx}{\int_0^{h'} \int_{\psi(y)}^{\phi(y)} dy \, dx}, \qquad \bar{y} = \frac{\int_0^{h'} \int_{\psi(y)}^{\phi(y)} y \, dy \, dx}{\int_0^{h'} \int_{\psi(y)}^{\phi(y)} dy \, dx},$$

where $\psi(y) = y\cot\beta$, $\phi(y) = \sqrt{(a^2 - y^2)}$, $h' = a\sin\beta$.

The integrations may be easily effected.

If we wish to integrate with respect to y first, we shall have to divide the figure into two parts by a straight line drawn from E perpendicular to OB. For the part to the left of the dividing line the limits of y are 0 and $x \tan\beta$, and

those of x are 0 and $a\cos\beta$. For the part to the right of the dividing line the limits of y are 0 and $\sqrt{(a^2-x^2)}$, and those of x are $a\cos\beta$ and a. Hence

$$\bar{x} = \frac{\int_0^{a\cos\beta}\int_0^{x\tan\beta} x\,dx\,dy + \int_{a\cos\beta}^{a}\int_0^{\sqrt{(a^2-x^2)}} x\,dx\,dy}{\int_0^{a\cos\beta}\int_0^{x\tan\beta} dx\,dy + \int_{a\cos\beta}^{a}\int_0^{\sqrt{(a^2-x^2)}} dx\,dy}.$$

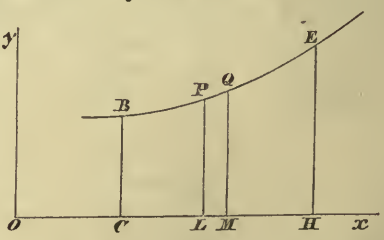
Similarly $\bar{y}$ may be expressed.

We have treated this example as an illustration of integration rather than for the purpose of obtaining the result in the simplest form. We might proceed thus; the centre of gravity must lie on the straight line which bisects the angle EOB. Hence taking this straight line for the initial line and using polar co-ordinates, we have $\bar{y} = 0$, and

$$\bar{x} = \frac{\int_0^{a}\int_{-\frac{1}{2}\beta}^{\frac{1}{2}\beta} r^2\cos\theta\,dr\,d\theta}{\int_0^{a}\int_{-\frac{1}{2}\beta}^{\frac{1}{2}\beta} r\,dr\,d\theta} = \frac{4a\sin\frac{1}{2}\beta}{3\beta}.$$

Solid of Revolution.

124. Let a surface be generated by the revolution of the curve $BPQE$ round the axis of x, and suppose we require the centre of gravity of a portion of the solid bounded by this surface and by planes perpendicular to the axis of revolution.



Let the co-ordinates of a point P in the curve be x and y, and $x+\Delta x$ the abscissa of an adjacent point Q. As the curve revolves round the axis of x, the area $PQML$ will generate a volume which is ultimately equal to $\pi y^2\Delta x$. Also the abscissa of its centre of gravity will be x ultimately. Hence

$$\bar{x} = \frac{\int \pi y^2 x\,dx}{\int \pi y^2\,dx} = \frac{\int y^2 x\,dx}{\int y^2\,dx}.$$

The centre of gravity of the solid is obviously in the straight line Ox, so that we only require the value of $\bar{x}$ in order to determine its position.

125. Ex. 1. Let it be required to find the centre of gravity of a portion of a paraboloid. Suppose $y^2 = 4ax$ the equation to the generating parabola, and that the solid is bounded by planes distant c and h respectively from the vertex; then

$$\bar{x} = \frac{\int_c^h 4ax^2 dx}{\int_c^h 4ax dx} = \frac{2}{3} \cdot \frac{h^3 - c^3}{h^2 - c^2}.$$

If we put $c = 0$ we find for the centre of gravity of a segment of a paraboloid commencing at the vertex

$$\bar{x} = \frac{2h}{3}.$$

Ex. 2. Required the centre of gravity of a portion of a sphere intercepted between two parallel planes.

Let $y^2 = a^2 - x^2$ be the equation to the generating circle;

$$\bar{x} = \frac{\int_c^h (a^2 - x^2)\, x dx}{\int_c^h (a^2 - x^2)\, dx} = \frac{\frac{1}{2}a^2(h^2 - c^2) - \frac{1}{4}(h^4 - c^4)}{a^2(h - c) - \frac{1}{3}(h^3 - c^3)}.$$

If we put $c = 0$ and $h = a$, we find for the centre of gravity of a hemisphere

$$\bar{x} = \tfrac{3}{8}a.$$

Ex. 3. Find the centre of gravity of the solid generated by the revolution of the cycloid $y = \sqrt{(2ax - x^2)} + a \operatorname{vers}^{-1} \frac{x}{a}$ round the axis of x.

Here $$\bar{x} = \frac{\int_0^{2a} y^2 x dx}{\int_0^{2a} y^2 dx}.$$

Now $$y^2 = 2ax - x^2 + 2a\sqrt{(2ax - x^2)} \operatorname{vers}^{-1} \frac{x}{a} + a^2 \left(\operatorname{vers}^{-1} \frac{x}{a} \right)^2.$$

Thus the numerator of $\bar{x}$ consists of three integrals of which we will give the values; these values may be obtained without difficulty by transforming the integrals where $\operatorname{vers}^{-1} \frac{x}{a}$

occurs by the assumption $\text{vers}^{-1}\frac{x}{a}=\theta$, so that $x=a(1-\cos\theta)$, and then integrating by parts. We shall find

$$\int_0^{2a}(2ax-x^2)\,x\,dx=\frac{4a^4}{3},$$

$$2a\int_0^{2a}x\sqrt{(2ax-x^2)}\,\text{vers}^{-1}\frac{x}{a}\,dx=2a\left(\frac{4a^3}{9}+\frac{\pi^2a^3}{4}\right),$$

$$a^2\int_0^{2a}x\left(\text{vers}^{-1}\frac{x}{a}\right)^2dx=\left(\frac{5\pi^2}{4}-4\right)a^4.$$

Hence the numerator of $\bar{x}$ is $\left(\frac{7\pi^2}{4}-\frac{16}{9}\right)a^4$.

Also the denominator of $\bar{x}$ consists of three integrals which have the following values,

$$\int_0^{2a}(2ax-x^2)\,dx=\frac{4a^3}{3},$$

$$2a\int_0^{2a}\sqrt{(2ax-x^2)}\,\text{vers}^{-1}\frac{x}{a}\,dx=2a\,\frac{\pi^2a^2}{4},$$

$$a^2\int_0^{2a}\left(\text{vers}^{-1}\frac{x}{a}\right)^2dx=(\pi^2-4)\,a^3.$$

Hence the denominator of $\bar{x}$ is $\left(\frac{3\pi^2}{2}-\frac{8}{3}\right)a^3$.

Therefore $$\bar{x}=\frac{\left(\frac{7\pi^2}{4}-\frac{16}{9}\right)a^4}{\left(\frac{3\pi^2}{2}-\frac{8}{3}\right)a^3}=\frac{(63\pi^2-64)\,a}{6\,(9\pi^2-16)}.$$

126. If a solid of revolution be formed by revolving a curve round the axis of y, we find for the position of the centre of gravity

$$\bar{y}=\frac{\int\pi x^2y\,dy}{\int\pi x^2dy}=\frac{\int x^2y\,dy}{\int x^2dy}.$$

For example, let the cycloid $y=\sqrt{(2ax-x^2)}+a\,\text{vers}^{-1}\frac{x}{a}$ revolve round the axis of y, and suppose that we require the

centre of gravity of the volume generated by that half of the curve for which y is positive. Here

$$\bar{y} = \frac{\int_0^{\pi a} x^2 y dy}{\int_0^{\pi a} x^2 dy} .$$

Now $\int x^2 y dy = \int x^2 y \frac{dy}{dx} dx$; thus in the present case,

$$\int_0^{\pi a} x^2 y dy = \int_0^{2a} x^2 y \frac{dy}{dx} dx.$$

Similarly $$\int_0^{\pi a} x^2 dy = \int_0^{2a} x^2 \frac{dy}{dx} dx.$$

Thus $$\bar{y} = \frac{\int_0^{2a} x^2 y \left(\frac{2a-x}{x}\right)^{\frac{1}{2}} dx}{\int_0^{2a} x^2 \left(\frac{2a-x}{x}\right)^{\frac{1}{2}} dx} = \frac{\int_0^{2a} yx \sqrt{(2ax-x^2)}\, dx}{\int_0^{2a} x \sqrt{(2ax-x^2)}\, dx} .$$

The numerator of $\bar{y}$ consists of two integrals which have the following values,

$$\int_0^{2a} x(2ax-x^2)\, dx = \frac{4a^4}{3} ,$$

$$a \int_0^{2a} x \sqrt{(2ax-x^2)} \operatorname{vers}^{-1} \frac{x}{a} dx = a \left(\frac{4a^3}{9} + \frac{\pi^2 a^3}{4}\right) .$$

The value of the denominator of $\bar{y}$ is $\frac{\pi}{2} a^3$.

Therefore $$\bar{y} = \frac{\frac{4a^4}{3} + \frac{4a^4}{9} + \frac{\pi^2 a^4}{4}}{\frac{\pi}{2} a^3} = \left(\frac{16}{9} + \frac{\pi^2}{4}\right) \frac{2a}{\pi} .$$

127. We may also find it convenient in some cases to use formulæ involving double integration.

Suppose the figure in Art. 118 to revolve round the axis of x; let x, y be the co-ordinates of s; and $x+\Delta x$, $y+\Delta y$ those of t. The area st generates by revolution an elementary ring, the volume of which is $\pi (y+\Delta y)^2 \Delta x - \pi y^2 \Delta x$; this may be put ultimately equal to $2\pi y \Delta y \Delta x$. The centre of gravity of this ring is on the axis of x, and its abscissa is

ultimately x. Hence by proceeding as before we shall have ultimately

$$\bar{x} = \frac{\int_c^h \int_{\psi(x)}^{\phi(x)} yx\,dx\,dy}{\int_c^h \int_{\psi(x)}^{\phi(x)} y\,dx\,dy},$$

where $y = \psi(x)$ is the equation to the lower bounding curve and $y = \phi(x)$ to the upper, and c and h are the abscissæ of the planes which bound the solid of revolution perpendicularly to its axis.

Similarly, if the solid is formed by revolving the area included between two curves round the axis of y, we shall have

$$\bar{y} = \frac{\int_{c'}^{h'} \int_{\psi(y)}^{\phi(y)} xy\,dy\,dx}{\int_{c'}^{h'} \int_{\psi(y)}^{\phi(y)} x\,dy\,dx}.$$

Or we may use polar formulæ. Suppose the figure in Art. 122 to revolve round the axis of x; let r, θ be the polar coordinates of s; and $r + \Delta r$, $\theta + \Delta\theta$, those of t. The volume of the ring generated by the revolution of the area st is ultimately $2\pi r \sin\theta r \Delta r \Delta\theta$; and the abscissa of the centre of gravity of the ring is ultimately $r\cos\theta$. Hence

$$\bar{x} = \frac{\iint r^3 \sin\theta\cos\theta\,d\theta\,dr}{\iint r^2 \sin\theta\,d\theta\,dr}.$$

Similarly, if the figure revolve round the axis of y

$$\bar{y} = \frac{\iint r^3 \cos\theta\sin\theta\,d\theta\,dr}{\iint r^2 \cos\theta\,d\theta\,dr}.$$

We have hitherto assumed the solid of revolution to be of *uniform* density; if this be not the case the formulæ must be modified. For example, take the first formula in the present Article; suppose that ρ denotes the density at the point (x, y). Then the mass of the ring considered will be ultimately $2\pi\rho y \Delta y \Delta x$. Hence

$$\bar{x} = \frac{\int_c^h \int_{\psi(x)}^{\phi(x)} \rho y x \, dx \, dy}{\int_c^h \int_{\psi(x)}^{\phi(x)} \rho y \, dx \, dy}.$$

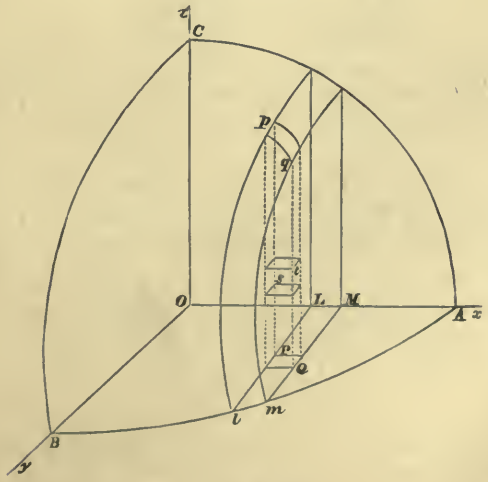
And ρ being supposed a known function of x and y, the integrations present no theoretical difficulty.

Similarly the polar formulæ may be modified. For example, instead of the formula given above for $\bar{x}$ we now obtain

$$\bar{x} = \frac{\iint \rho r^3 \sin\theta \cos\theta \, d\theta \, dr}{\iint \rho r^2 \sin\theta \, d\theta \, dr}.$$

In this case ρ must be expressible as a function of r and θ, in order that the integrations may be practicable. The most common cases are two; in one the density depends only on the distance from a fixed point in the axis of revolution, so that by taking this point as origin ρ is a function of r; in the other case the density depends only on the distance from the axis of revolution, so that ρ is a function of $r \sin\theta$.

Any Solid.



128. To find the centre of gravity of a solid we divide it into elements as follows: draw a series of planes perpen-

dicular to the axis of x, then two consecutive planes will include between them a slice such as $LpkmqM$ in the figure; draw a second series of planes perpendicular to the axis of y, then each slice is divided into strips such as $PpqQ$ in the figure; lastly, draw planes perpendicular to the axis of z, then each strip is divided into parallelepipeds such as st in the figure. Let x, y, z be the co-ordinates of s, and $x+\Delta x$, $y+\Delta y$, $z+\Delta z$ those of t; then $\Delta x\Delta y\Delta z$ is the volume of st, and as the co-ordinates of its centre of gravity are ultimately x, y, and z, we have

$$\bar{x}=\frac{\iiint x\,dx\,dy\,dz}{\iiint dx\,dy\,dz},\quad \bar{y}=\frac{\iiint y\,dx\,dy\,dz}{\iiint dx\,dy\,dz},\quad \bar{z}=\frac{\iiint z\,dx\,dy\,dz}{\iiint dx\,dy\,dz}.$$

When the axes are oblique, let αx, βy, γz denote the perpendicular distances of the point (x, y, z) from the co-ordinate planes. Then α, β, γ will be constants, being the sines of the angles which the axes make with the co-ordinate planes. Also if $k\Delta x\Delta y\Delta z$ denote the volume of the parallelepiped whose edges are Δx, Δy, Δz, k will be constant. The perpendicular distance of the centre of gravity from the plane of (y, z) will be $\alpha\bar{x}$ and will also be

$$\frac{\int\alpha x\,k\,dx\,dy\,dz}{\int k\,dx\,dy\,dz}=\alpha\frac{\int x\,dx\,dy\,dz}{\int dx\,dy\,dz};$$

hence we have $\bar{x}=\dfrac{\int x\,dx\,dy\,dz}{\int dx\,dy\,dz}$, and similar reasoning applies to the values of $\bar{y}$ and $\bar{z}$.

129. In applying the above formulæ to examples, great care is necessary in assigning proper limits to the integrations; this we shall illustrate by Examples.

Ex. 1. Find the centre of gravity of the eighth part of an ellipsoid cut off by three principal planes.

Let the equation to the surface be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

Then the equation to the curve in which the surface meets the plane of (x, y) is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

Integrate first with respect to z, and take for the limits $z=0$ and $z=c\sqrt{\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)}$; we thus include all the elements like st which form the strip $PpqQ$. Next integrate with respect to y, and take for the limits $y=0$ and $y=b\sqrt{\left(1-\frac{x^2}{a^2}\right)}$; we thus include all the strips like $PpqQ$ which form the slice $LplmqM$. Lastly integrate with respect to x, and take for the limits $x=0$ and $x=a$; we thus include all the slices like $LplmqM$ which form the solid we are considering. Hence

$$\bar{x}=\frac{\int_0^a\int_0^{y_1}\int_0^{z_1}x\,dx\,dy\,dz}{\int_0^a\int_0^{y_1}\int_0^{z_1}dx\,dy\,dz},$$

where we put z_1 for $c\sqrt{\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)}$,

and y_1 for $b\sqrt{\left(1-\frac{x^2}{a^2}\right)}$.

Now $$\int_0^{z_1}dz=z_1=c\sqrt{\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)};$$

therefore $$\bar{x}=\frac{\int_0^a\int_0^{y_1}x\sqrt{\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)}\,dx\,dy}{\int_0^a\int_0^{y_1}\sqrt{\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)}\,dx\,dy}.$$

And $\int_0^{y_1}\sqrt{\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)}\,dy$, or $\frac{1}{b}\int_0^{y_1}\sqrt{(y_1^2-y^2)}\,dy=\frac{\pi y_1^2}{4b}$;

therefore $$\bar{x}=\frac{\int_0^a xy_1^2dx}{\int_0^a y_1^2dx}=\frac{\int_0^a\left(1-\frac{x^2}{a^2}\right)x\,dx}{\int_0^a\left(1-\frac{x^2}{a^2}\right)dx}=\frac{3a}{8}.$$

Similarly $$\bar{y}=\frac{3b}{8},\quad \bar{z}=\frac{3c}{8}.$$

We may in this example effect the integrations with equal

simplicity in any order we please; if we integrate first for x, then for y, and lastly for z, we shall have

$$\bar{x} = \frac{\int_0^c \int_0^{y_1} \int_0^{x_1} x\,dz\,dy\,dx}{\int_0^c \int_0^{y_1} \int_0^{x_1} dz\,dy\,dx};$$

where x_1 stands for $a\sqrt{\left(1 - \frac{z^2}{c^2} - \frac{y^2}{b^2}\right)}$,

and y_1 stands for $b\sqrt{\left(1 - \frac{z^2}{c^2}\right)}$.

This will be easily seen by drawing a figure so as to make the planes bounding the *slice* parallel to that of (x, y), and the edges of the *strip* parallel to the axis of x.

Ex. 2. Let it be required to find the centre of gravity of the solid bounded by the planes $z = \beta x$, $z = \gamma x$, and the cylinder $y^2 = 2ax - x^2$. We shall have

$$\bar{x} = \frac{\int_0^{2a} \int_{-y_1}^{y_1} \int_{\beta x}^{\gamma x} x\,dx\,dy\,dz}{\int_0^{2a} \int_{-y_1}^{y_1} \int_{\beta x}^{\gamma x} dx\,dy\,dz},$$

where y_1 is put for $\sqrt{(2ax - x^2)}$.

Now $$\int_{\beta x}^{\gamma x} dz = (\gamma - \beta)\, x,$$

therefore $$\bar{x} = \frac{\int_0^{2a} \int_{-y_1}^{y_1} x^2\,dx\,dy}{\int_0^{2a} \int_{-y_1}^{y_1} x\,dx\,dy}.$$

Also $$\int_{-y_1}^{y_1} dy = 2\sqrt{(2ax - x^2)};$$

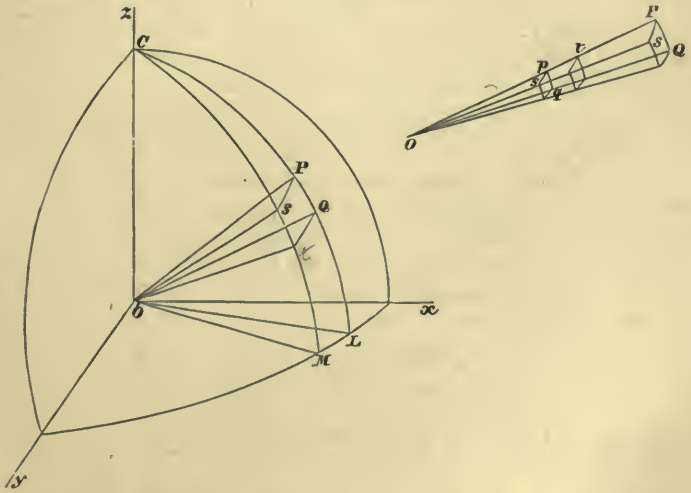
therefore
$$\bar{x} = \frac{\int_0^{2a} x^2 \sqrt{(2ax - x^2)}\, dx}{\int_0^{2a} x \sqrt{(2ax - x^2)}\, dx} = \frac{5a}{4}.$$

(See *Integral Calculus*, Ex. 5 to Chap. III.)

Similarly we may find
$$\bar{y} = 0, \ \bar{z} = \frac{5a(\beta + \gamma)}{8}.$$

130. It is often convenient to divide a solid into polar elements.

Let a series of planes be drawn through the axis of z; the solid is thus divided into wedge-shaped slices such as $COML$. Let a series of right cones be described round the axis of z having their vertices at O; thus each slice is divided into pyramidal solids like $OPQS$. Lastly, let a series of concentric



spheres be described round O as centre; thus each pyramid is divided into elements similar to $pqst$.

Let $xOL = \phi$, $COP = \theta$, $Op = r$,
$LOM = \Delta\phi$, $POQ = \Delta\theta$, $pt = \Delta r$.

Then pq is the arc of a circle of which the radius is r and the angle $\Delta\theta$; therefore $pq = r\Delta\theta$.

Also ps is the arc of a circle of which the radius is $r\sin\theta$ and the angle $\Delta\phi$; therefore $ps = r\sin\theta\Delta\phi$.

Hence, since the element $pqst$ is ultimately a parallelepiped, its volume is $r^2\sin\theta\Delta\theta\Delta\phi\Delta r$.

Also the co-ordinates of its centre of gravity are ultimately $r\cos\phi\sin\theta$, $r\sin\phi\sin\theta$, and $r\cos\theta$. Hence supposing its density to be ρ, we have

$$\bar{x} = \frac{\iiint \rho r^3 \sin^2\theta\cos\phi\, d\phi\, d\theta\, dr}{\iiint \rho r^2 \sin\theta\, d\phi\, d\theta\, dr},$$

$$\bar{y} = \frac{\iiint \rho r^3 \sin^2\theta\sin\phi\, d\phi\, d\theta\, dr}{\iiint \rho r^2 \sin\theta\, d\phi\, d\theta\, dr},$$

$$\bar{z} = \frac{\iiint \rho r^3 \sin\theta\cos\theta\, d\phi\, d\theta\, dr}{\iiint \rho r^2 \sin\theta\, d\phi\, d\theta\, dr}.$$

131. Ex. 1. Apply the preceding formulæ to find the centre of gravity of a hemisphere whose density varies as the n^{th} power of the distance from the centre.

Take the axis of z perpendicular to the plane base of the hemisphere. Let a be the radius of the hemisphere, and $\rho = \mu r^n$, where μ is a constant. First integrate with respect to r from 0 to a; we thus include all the elements like $pqst$ comprised in the pyramid $OPQS$. Next integrate with respect to θ from 0 to $\frac{1}{2}\pi$, we thus include all the pyramids in the slice $COML$. Finally, integrate from $\phi = 0$ to $\phi = 2\pi$; we thus include all the slices. Thus

$$\bar{z} = \frac{\int_0^{2\pi}\int_0^{\frac{1}{2}\pi}\int_0^a r^{n+3}\sin\theta\cos\theta\, d\phi\, d\theta\, dr}{\int_0^{2\pi}\int_0^{\frac{1}{2}\pi}\int_0^a r^{n+2}\sin\theta\, d\phi\, d\theta\, dr},$$

$$= \frac{n+3}{n+4}\,a\,\frac{\int_0^{2\pi}\int_0^{\frac{1}{2}\pi}\sin\theta\cos\theta\, d\phi\, d\theta}{\int_0^{2\pi}\int_0^{\frac{1}{2}\pi}\sin\theta\, d\phi\, d\theta} = \frac{n+3}{n+4}\cdot\frac{a}{2};$$

$\bar{x}$ and $\bar{y}$ each $= 0$.

Ex. 2. A right cone has its vertex on the surface of a sphere and its axis coincident with a diameter of the sphere, find the centre of gravity of the portion of the cone bounded

by the sphere. Take the axis of z coincident with that of the cone; suppose a the radius of the sphere, β the semi-vertical angle of the cone. The polar equation to the sphere is $r = 2a\cos\theta$, and to the cone $\theta = \beta$. Hence we have

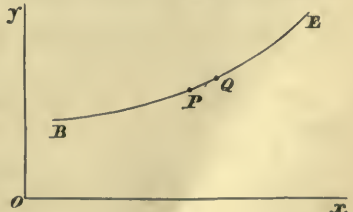
$$\bar{z} = \frac{\int_0^{2\pi}\int_0^{\beta}\int_0^{2a\cos\theta} r^3 \cos\theta \sin\theta \, d\phi \, d\theta \, dr}{\int_0^{2\pi}\int_0^{\beta}\int_0^{2a\cos\theta} r^2 \sin\theta \, d\phi \, d\theta \, dr};$$

$\bar{x}$ and $\bar{y}$ each $= 0$.

Curve.

132. Suppose a very small circle of variable radius to move so that its centre describes a given curve and its plane is always perpendicular to the tangent line of the curve, we may require the centre of gravity of the solid generated. The simplest case is that in which the radius is constant and the solid of uniform density; the result then depends *solely* on the nature of the curve described by the centre of the circle, and for shortness the process is called finding *the centre of gravity of a curve.*

Let $BPQE$ be a *plane* curve; BP the length measured from some fixed point B, $BP = s$, $PQ = \Delta s$; x, y the co-ordinates of P. Let k denote the area of a transverse section; then the volume of the element PQ is $k\Delta s$, and the co-ordinates of its centre of gravity are ultimately x and y. Hence



$$\bar{x} = \frac{\int kx\,ds}{\int k\,ds} = \frac{\int x\,ds}{\int ds} \text{ if } k \text{ be constant} \quad \ldots\ldots(1),$$

$$\bar{y} = \frac{\int ky\,ds}{\int k\,ds} = \frac{\int y\,ds}{\int ds} \quad \ldots\ldots(2).$$

Since $\frac{ds}{dx} = \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}$, we may also write

$$\bar{x} = \frac{\int x \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}} dx}{\int \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}} dx}, \quad \bar{y} = \frac{\int y \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}} dx}{\int \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}} dx} \quad \ldots(3).$$

From the equation to the curve y and $\frac{dy}{dx}$ are known in terms of x; their values must be substituted in the preceding expressions and the integrations then effected.

If we use polar co-ordinates we have $x = r\cos\theta$, $y = r\sin\theta$, and $\frac{ds}{d\theta} = \sqrt{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}}$.

Hence

$$\bar{x} = \frac{\int r\cos\theta \sqrt{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}} d\theta}{\int \sqrt{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}} d\theta}, \quad \bar{y} = \frac{\int r\sin\theta \sqrt{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}} d\theta}{\int \sqrt{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}} d\theta} \quad \ldots\ldots(4);$$

for r and $\frac{dr}{d\theta}$ we must substitute their values in terms of θ given by the equation to the curve.

133. Ex. 1. A straight rod of uniform thickness and density.

Taking the origin on the line we have $y = \beta x$, where β is constant; hence, by equations (3) of Art. 132, supposing the origin to be at one end of the rod and h the abscissa of the other end,

$$\bar{x} = \frac{\int_0^h x\,dx}{\int_0^h dx} = \frac{h}{2}, \qquad \bar{y} = \frac{\beta\int_0^h x\,dx}{\int_0^h dx} = \frac{\beta h}{2}.$$

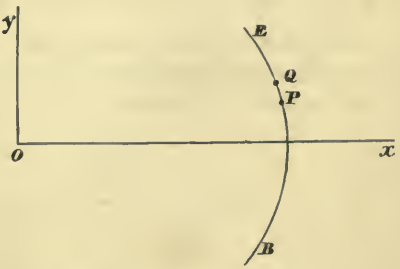
That is, the centre of gravity is the middle point of the rod.

Ex. 2. Suppose the transverse section of the rod to vary as the n^{th} power of the distance from one end. Take the origin at this end, and suppose the axis of x to coincide with the axis of the rod; then $\bar{y} = 0$, and in equation (1) of Art. 132 we put μx^n for k, where μ is constant. Hence, if h be the length of the rod,

$$\bar{x} = \frac{\int_0^h x^{n+1}\, ds}{\int_0^h x^n\, ds} = \frac{\int_0^h x^{n+1}\, dx}{\int_0^h x^n\, dx} = \frac{n+1}{n+2}\, h.$$

Ex. 3. An arc of a circle.

Take the origin at the centre of the circle, and the axis of x bisecting the arc. Then $\bar{y} = 0$; and supposing 2α to be the angle subtended at O by the given arc, and a the radius of the circle, we have, by Art. 132, equation (4),



$$\bar{x} = \frac{a^2 \int_{-\alpha}^{\alpha} \cos\theta\, d\theta}{a \int_{-\alpha}^{\alpha} d\theta} = \frac{a \sin\alpha}{\alpha}.$$

Ex. 4. The arc of a semicycloid.

Take the origin at the vertex, and the axis of y a tangent there; then $\left(\frac{dy}{dx}\right)^2 = \frac{2a - x}{x}$: hence

$$\bar{x} = \frac{\int_0^{2a} x \sqrt{\left(\frac{2a}{x}\right)}\, dx}{\int_0^{2a} \sqrt{\left(\frac{2a}{x}\right)}\, dx} = \frac{\int_0^{2a} x^{\frac{1}{2}}\, dx}{\int_0^{2a} x^{-\frac{1}{2}}\, dx} = \frac{\frac{2}{3}(2a)^{\frac{3}{2}}}{2\,(2a)^{\frac{1}{2}}} = \frac{2a}{3},$$

$$\bar{y} = \frac{\int_0^{2a} y \sqrt{\left(\frac{2a}{x}\right)}\, dx}{\int_0^{2a} \sqrt{\left(\frac{2a}{x}\right)}\, dx} = \frac{\int_0^{2a} \frac{y}{\sqrt{x}}\, dx}{\int_0^{2a} \frac{dx}{\sqrt{x}}}.$$

Now $\int \frac{y}{\sqrt{x}}\,dx = 2\int y \frac{d\sqrt{x}}{dx}\,dx = 2y\sqrt{x} - 2\int \sqrt{x}\,\frac{dy}{dx}\,dx$

$= 2y\sqrt{x} - 2\int \sqrt{(2a-x)}\,dx = 2y\sqrt{x} + \frac{4}{3}(2a-x)^{\frac{3}{2}}$;

therefore $\int_0^{2a} \frac{y}{\sqrt{x}}\,dx = 2\pi a\,(2a)^{\frac{1}{2}} - \frac{4}{3}(2a)^{\frac{3}{2}}$;

therefore $\bar{y} = \frac{2\pi a\,(2a)^{\frac{1}{2}} - \frac{4}{3}(2a)^{\frac{3}{2}}}{2\,(2a)^{\frac{1}{2}}} = (\pi - \tfrac{4}{3})\,a.$

Ex. 5. The curve $y = \frac{1}{2}c\,(e^{\frac{x}{c}} + e^{-\frac{x}{c}})$.

If s' denote the length of an arc of the curve measured from the point whose co-ordinates are 0, c, to the point (x', y'), we have for the co-ordinates of its centre of gravity

$$\bar{x} = \frac{\int_0^{x'} x \frac{ds}{dx}\,dx}{s'}, \qquad \bar{y} = \frac{\int_0^{x'} y \frac{ds}{dx}\,dx}{s'}.$$

Now $\frac{dy}{dx} = \frac{1}{2}(e^{\frac{x}{c}} - e^{-\frac{x}{c}})$,

therefore $1 + \left(\frac{dy}{dx}\right)^2 = \frac{1}{4}(e^{\frac{x}{c}} + e^{-\frac{x}{c}})^2$,

and $\frac{ds}{dx} = \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}} = \frac{1}{2}(e^{\frac{x}{c}} + e^{-\frac{x}{c}})$;

thus $s = \frac{c}{2}(e^{\frac{x}{c}} - e^{-\frac{x}{c}})$, and $s' = \frac{c}{2}(e^{\frac{x'}{c}} - e^{-\frac{x'}{c}})$.

Also $\int x \frac{ds}{dx}\,dx = \frac{1}{2}\int x\,(e^{\frac{x}{c}} + e^{-\frac{x}{c}})\,dx$

$= \frac{cx}{2}(e^{\frac{x}{c}} - e^{-\frac{x}{c}}) - \frac{c}{2}\int (e^{\frac{x}{c}} - e^{-\frac{x}{c}})\,dx$

$= \frac{cx}{2}(e^{\frac{x}{c}} - e^{-\frac{x}{c}}) - \frac{c^2}{2}(e^{\frac{x}{c}} + e^{-\frac{x}{c}})$;

therefore $$\int_0^{x'} x\frac{ds}{dx}dx = \frac{cx'}{2}(e^{\frac{x'}{c}} - e^{-\frac{x'}{c}}) - \frac{c^2}{2}(e^{\frac{x'}{c}} + e^{-\frac{x'}{c}}) + c^2$$

$$= x's' - cy' + c^2,$$

and $$\bar{x} = x' - \frac{c(y'-c)}{s'}.$$

Also $$\int y\frac{ds}{dx}dx = \frac{c}{4}\int(e^{\frac{x}{c}} + e^{-\frac{x}{c}})(e^{\frac{x}{c}} + e^{-\frac{x}{c}})\,dx$$

$$= \frac{c}{4}\int(e^{\frac{2x}{c}} + 2 + e^{-\frac{2x}{c}})\,dx = \frac{c^2}{8}(e^{\frac{2x}{c}} - e^{-\frac{2x}{c}}) + \frac{cx}{2};$$

therefore $$\int_0^{x'} y\frac{ds}{dx}dx = \frac{c^2}{8}(e^{\frac{2x'}{c}} - e^{-\frac{2x'}{c}}) + \frac{cx'}{2}$$

$$= \frac{y's'}{2} + \frac{cx'}{2},$$

and $$\bar{y} = \frac{y'}{2} + \frac{cx'}{2s'}.$$

134. If the curve be not plane, the formulæ (1) and (2) of Art. 132 still hold; in order to effect the integrations we may use the formula

$$\frac{ds}{dz} = \sqrt{\left\{1 + \left(\frac{dx}{dz}\right)^2 + \left(\frac{dy}{dz}\right)^2\right\}};$$

and from the two equations to the curve we must find $\frac{dx}{dz}$ and $\frac{dy}{dz}$ in terms of z. (See *Integral Calculus*, Art. 120.) For example, in the helix

$$x = a\cos nz, \quad y = a\sin nz;$$

therefore $$\frac{ds}{dz} = \sqrt{(1 + n^2a^2)},$$

$$\bar{x} = \frac{\int\sqrt{(1+n^2a^2)}\,xdz}{\int\sqrt{(1+n^2a^2)}\,dz} = \frac{\int a\cos nz\,dz}{\int dz}.$$

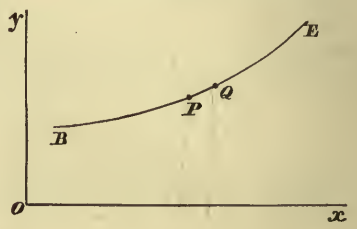
If we take for the limits $z = 0$ and $z = h$, we have

$$\bar{x} = \frac{a \sin nh}{nh}.$$

Similarly
$$\bar{y} = \frac{a(1 - \cos nh)}{nh}, \quad \bar{z} = \tfrac{1}{2}h.$$

Surface of Revolution.

135. Let $BPQE$ be a curve which by revolving round the axis of x generates a surface. Suppose a shell of which this surface is the exterior boundary, and of which the interior boundary is another surface of revolution round the axis of x indefinitely near to the former. Required the centre of gravity of a portion of this shell cut off by planes perpendicular to the axis of x.



Let P, Q be adjacent points in the exterior generating curve; suppose B a fixed point in the curve, let $BP = s$, and $PQ = \Delta s$; let x, y be the co-ordinates of P; k the thickness of the shell at P. The volume of the element contained between two planes perpendicular to the axis of x through P and Q respectively is ultimately $2\pi yk\Delta s$, and the abscissa of the centre of gravity of this element is ultimately x; hence

$$\bar{x} = \frac{\int 2\pi ykx ds}{\int 2\pi yk ds} = \frac{\int yx ds}{\int y ds},$$

if k be constant.

Since $\dfrac{ds}{dx} = \sqrt{\left\{1 + \left(\dfrac{dy}{dx}\right)^2\right\}}$, we have

$$\bar{x} = \frac{\int_c^h yx \sqrt{\left\{1 + \left(\dfrac{dy}{dx}\right)^2\right\}} dx}{\int_c^h y \sqrt{\left\{1 + \left(\dfrac{dy}{dx}\right)^2\right\}} dx},$$

where c and h are the distances of the bounding planes from the origin.

Since the centre of gravity required is on the axis of x, we need only the value of $\bar{x}$ in order to determine its position.

Similarly, if the curve $BPQE$ generates a surface by revolving round the axis of y, we have

$$\bar{y} = \frac{\int_c^h xy \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}\, dx}{\int_c^h x \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}\, dx},$$

where c and h denote as before the abscissæ of the extremities of the curve.

If we use polar co-ordinates, we have $x = r\cos\theta$, $y = r\sin\theta$, and

$$\frac{ds}{d\theta} = \sqrt{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}};$$

thus if the curve revolves round the axis of x, we have

$$\bar{x} = \frac{\int r^2 \sin\theta\cos\theta \sqrt{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}}\, d\theta}{\int r\sin\theta \sqrt{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}}\, d\theta},$$

and if the curve revolves round the axis of y, we have

$$\bar{y} = \frac{\int r^2 \cos\theta\sin\theta \sqrt{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}}\, d\theta}{\int r\cos\theta \sqrt{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}}\, d\theta}.$$

The limits of the integrations are the values of θ which correspond to the extremities of the curve.

Ex. 1. A cylindrical surface.

Take the axis of the cylinder as the axis of x; then $y =$ the radius of the cylinder, and is constant; hence

$$\bar{x} = \frac{\int_c^h x dx}{\int_c^h dx} = \frac{\frac{1}{2}(h^2 - c^2)}{h - c} = \frac{h + c}{2}.$$

Ex. 2. A spherical surface.

Here
$$y = \sqrt{(a^2 - x^2)},$$
$$\frac{dy}{dx} = -\frac{x}{\sqrt{(a^2 - x^2)}},$$
$$\frac{ds}{dx} = \frac{a}{\sqrt{(a^2 - x^2)}} = \frac{a}{y};$$

therefore
$$\bar{x} = \frac{\int_c^h axdx}{\int_c^h adx} = \frac{c+h}{2}.$$

Hence in both these examples the centre of gravity is equidistant from the two bounding planes.

Ex. 3. The surface of a cone.

Here $y = x \tan \alpha$, where α is the semivertical angle,

$$\frac{ds}{dx} = \sec \alpha,$$

$$\bar{x} = \frac{\int_c^h x \tan \alpha x \sec \alpha \, dx}{\int_c^h x \tan \alpha \sec \alpha dx} = \frac{2(h^3 - c^3)}{3(h^2 - c^2)} = \frac{2(h^2 + hc + c^2)}{3(h + c)}.$$

Ex. 4. Suppose the cycloid

$$y = \sqrt{(2ax - x^2)} + a \operatorname{vers}^{-1} \frac{x}{a}$$

to revolve round the axis of x.

Here
$$\frac{dy}{dx} = \sqrt{\left(\frac{2a-x}{x}\right)}, \quad \frac{ds}{dx} = \sqrt{\left(\frac{2a}{x}\right)};$$

thus
$$\bar{x} = \frac{\int_0^{2a} yx \sqrt{\left(\frac{2a}{x}\right)} dx}{\int_0^{2a} y \sqrt{\left(\frac{2a}{x}\right)} dx} = \frac{\int_0^{2a} yx^{\frac{1}{2}} dx}{\int_0^{2a} yx^{-\frac{1}{2}} dx}.$$

Now
$$\int yx^{\frac{1}{2}} dx = \frac{2yx^{\frac{3}{2}}}{3} - \tfrac{2}{3} \int x^{\frac{3}{2}} \frac{dy}{dx} dx$$
$$= \frac{2yx^{\frac{3}{2}}}{3} - \tfrac{2}{3} \int x \sqrt{(2a - x)} dx;$$

therefore $$\int_0^{2a} yx^{\frac{1}{2}}dx = \frac{2\pi a\,(2a)^{\frac{3}{2}}}{3} - \tfrac{2}{3}\int_0^{2a} x\sqrt{(2a-x)}\,dx;$$

and $$\int x\sqrt{(2a-x)}\,dx = -\frac{2x(2a-x)^{\frac{3}{2}}}{3} + \tfrac{2}{3}\int(2a-x)^{\frac{3}{2}}dx$$

$$= -\frac{2x(2a-x)^{\frac{3}{2}}}{3} - \frac{4}{15}(2a-x)^{\frac{5}{2}},$$

therefore $$\int_0^{2a} x\sqrt{(2a-x)}\,dx = \frac{4}{15}(2a)^{\frac{5}{2}};$$

thus $$\int_0^{2a} yx^{\frac{1}{2}}dx = \frac{2\pi a(2a)^{\frac{3}{2}}}{3} - \frac{8}{45}(2a)^{\frac{5}{2}}.$$

Also $$\int_0^{2a} yx^{-\frac{1}{2}}\,dx = 2\pi a(2a)^{\frac{1}{2}} - \frac{4}{3}(2a)^{\frac{3}{2}},$$ (see page 154),

therefore $$\bar{x} = \frac{\dfrac{2\pi a(2a)^{\frac{3}{2}}}{3} - \dfrac{8}{45}(2a)^{\frac{5}{2}}}{2\pi a\,(2a)^{\frac{1}{2}} - \dfrac{4}{3}(2a)^{\frac{3}{2}}}$$

$$= a\,\frac{\dfrac{4\pi}{3} - \dfrac{32}{45}}{2\pi - \dfrac{8}{3}} = \frac{\dfrac{2a}{3}\left(\pi - \dfrac{8}{15}\right)}{\pi - \dfrac{4}{3}}.$$

Ex. 5. Suppose the cycloid

$$y = \sqrt{(2ax - x^2)} + a\,\text{vers}^{-1}\frac{x}{a}$$

to revolve round the axis of y, and that we require the centre of gravity of the surface generated by that half of the curve for which y is positive.

Here $$\bar{y} = \frac{\displaystyle\int_0^{2a} yx\sqrt{\left(\frac{2a}{x}\right)}dx}{\displaystyle\int_0^{2a} x\sqrt{\left(\frac{2a}{x}\right)}dx} = \frac{\displaystyle\int_0^{2a} yx^{\frac{1}{2}}dx}{\displaystyle\int_0^{2a} x^{\frac{1}{2}}dx}.$$

The value of the numerator was found in the preceding example; and

$$\int_0^{2a} x^{\frac{1}{2}}dx = \frac{2}{3}(2a)^{\frac{3}{2}},$$

therefore
$$\bar{y} = \frac{\frac{2\pi a}{3}(2a)^{\frac{3}{2}} - \frac{8}{45}(2a)^{\frac{5}{2}}}{\frac{2}{3}(2a)^{\frac{3}{2}}}$$
$$= a\left(\pi - \frac{8}{15}\right).$$

Ex. 6. Find the centre of gravity of the surface formed by revolving the curve $r = a(1 + \cos\theta)$ round the initial line. Here
$$\frac{dr}{d\theta} = -a\sin\theta, \quad r^2 + \left(\frac{dr}{d\theta}\right)^2 = 2a^2(1 + \cos\theta),$$

therefore
$$\frac{ds}{d\theta} = \sqrt{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}} = 2a\cos\frac{\theta}{2}.$$

Thus
$$\bar{x} = \frac{\int_0^\pi r^2 \sin\theta\cos\theta\, 2a\cos\frac{\theta}{2}\, d\theta}{\int_0^\pi r\sin\theta\, 2a\cos\frac{\theta}{2}\, d\theta}$$
$$= \frac{2a\int_0^\pi \cos^6\frac{\theta}{2}\left(2\cos^2\frac{\theta}{2} - 1\right)\sin\frac{\theta}{2}\, d\theta}{\int_0^\pi \cos^4\frac{\theta}{2}\sin\frac{\theta}{2}\, d\theta}.$$

Now
$$\int\cos^6\frac{\theta}{2}\left(2\cos^2\frac{\theta}{2} - 1\right)\sin\frac{\theta}{2}\, d\theta = -\frac{4}{9}\cos^9\frac{\theta}{2} + \frac{2}{7}\cos^7\frac{\theta}{2};$$

therefore
$$\int_0^\pi \cos^6\frac{\theta}{2}\left(2\cos^2\frac{\theta}{2} - 1\right)\sin\frac{\theta}{2}\, d\theta = \frac{4}{9} - \frac{2}{7}.$$

Similarly
$$\int_0^\pi \cos^4\frac{\theta}{2}\sin\frac{\theta}{2}\, d\theta = \frac{2}{5},$$

therefore
$$\bar{x} = \frac{2a\left(\frac{4}{9} - \frac{2}{7}\right)}{\frac{2}{5}} = \frac{50a}{63}.$$

Any Surface.

136. Let there be a shell having any given surface for one of its boundaries, and suppose its thickness indefinitely small. Let x, y, z be the co-ordinates of any point of the given surface, k the thickness at that point, ΔS the area of an element of the surface there, then $k\Delta S$ is ultimately the volume of this element, and x, y, z the co-ordinates of its centre of gravity; hence

$$\bar{x} = \frac{\int kx\,dS}{\int k\,dS},$$

and similar expressions hold for $\bar{y}$ and $\bar{z}$.

It may be shewn (see *Integral Calculus*, Art. 170) that if we take ΔS such that its projection on the plane of (x, y) is the rectangle $\Delta x \Delta y$,

$$\frac{\Delta S}{\Delta x \Delta y} = \sqrt{\left\{1 + \left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2\right\}} \text{ ultimately.}$$

Hence
$$\bar{x} = \frac{\iint kx \sqrt{\left\{1 + \left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2\right\}}\,dx\,dy}{\iint k \sqrt{\left\{1 + \left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2\right\}}\,dx\,dy}.$$

Ex. The surface of the eighth part of a sphere.

Here
$$x^2 + y^2 + z^2 = a^2,$$

$$\sqrt{\left\{1 + \left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2\right\}} = \frac{a}{\sqrt{(a^2 - x^2 - y^2)}}.$$

Hence
$$\bar{x} = \frac{\iint \frac{x\,dx\,dy}{\sqrt{(a^2 - x^2 - y^2)}}}{\iint \frac{dx\,dy}{\sqrt{(a^2 - x^2 - y^2)}}}.$$

First integrate with respect to y from $y = 0$ to $y = \sqrt{(a^2 - x^2)}$; we thus include all the elements that form the strip of surface of which $LlmM$ is the projection on the plane of (x, y); see fig. to Art. 128.

Now $$\int_0^{\sqrt{(a^2-x^2)}} \frac{dy}{\sqrt{(a^2-x^2-y^2)}} = \tfrac{1}{2}\pi;$$

therefore $$\bar{x} = \frac{\int \frac{1}{2}\pi x\, dx}{\int \frac{1}{2}\pi\, dx} = \frac{\int x\, dx}{\int dx}.$$

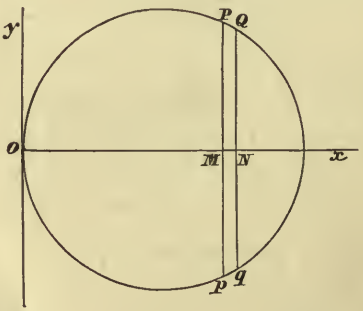
The limits of the integration for x are 0 and a;

therefore $$\bar{x} = \tfrac{1}{2}a.$$

Similarly $$\bar{y} = \tfrac{1}{2}a, \quad \bar{z} = \tfrac{1}{2}a.$$

137. In the preceding Articles we have given the usual formulæ for finding the centres of gravity of bodies, but particular cases may occur which may be most conveniently treated by special methods. We add some examples.

(1) A circle revolves round a tangent line through an



angle of 180°; find the centre of gravity of the solid generated. Let Oy be the tangent line about which the circle revolves, and let the plane of the paper bisect the solid; the centre of gravity will therefore lie in the axis of x. Let $OM = x$, $MP = y = \sqrt{(2ax - x^2)}$, $MN = \Delta x$. The figure $PQqp$ will by its revolution generate a semi-cylindrical shell, whose volume is ultimately $2y\pi x\Delta x$; the centre of gravity of this shell will be in the axis of x at a distance $\frac{2x}{\pi}$ from O (see Art. 133, Ex. 3);

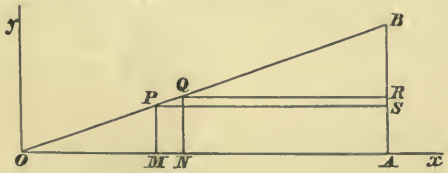
therefore
$$\bar{x} = \frac{\int_0^{2a} \frac{2x}{\pi} 2y\pi x\,dx}{\int_0^{2a} 2y\pi x\,dx} = \frac{\frac{2}{\pi}\int_0^{2a} yx^2\,dx}{\int_0^{2a} yx\,dx}$$

$$= \frac{2}{\pi} \frac{\int_0^{2a} x^2 \sqrt{(2ax - x^2)}\,dx}{\int_0^{2a} x\sqrt{(2ax - x^2)}\,dx}.$$

It will be found that $\bar{x} = \frac{5a}{2\pi}$.

(2) The density of a right cone varies as the n^{th} power of the distance from the axis; find the centre of gravity of the cone.

Let OAB be the right-angled triangle which by revolving



round Ox generates the cone. Let PS and QR be drawn parallel to the axis of x at distances y and $y + \Delta y$ respectively. Let

$$OA = h, \quad \text{angle } BOA = \alpha.$$

Then
$$OM = y \cot \alpha, \quad PS = h - y \cot \alpha.$$

The volume of the cylindrical shell generated by the revolution of $PQRS$ round Ox is ultimately

$$2\pi y\,\Delta y\,(h - y \cot \alpha).$$

Its density is μy^n, where μ is constant; therefore, its mass is

$$2\pi\mu y^{n+1}\,\Delta y\,(h - y \cot \alpha).$$

The distance of its centre of gravity from O is ultimately (see Art. 135, Ex. 1)

$$\tfrac{1}{2}(OM + OA), \text{ that is } \tfrac{1}{2}(h + y\cot\alpha);$$

therefore
$$\bar{x} = \frac{\int_0^{h\tan\alpha} 2\pi\mu y^{n+1}(h - y\cot\alpha)\,\tfrac{1}{2}(h + y\cot\alpha)\,dy}{\int_0^{h\tan\alpha} 2\pi\mu y^{n+1}(h - y\cot\alpha)\,dy}$$

$$= \frac{\frac{1}{2}\int_0^{h\tan\alpha} y^{n+1}(h^2 - y^2\cot^2\alpha)\,dy}{\int_0^{h\tan\alpha} y^{n+1}(h - y\cot\alpha)\,dy},$$

and the integrations can be easily performed.

(3) A shell has for its outer and inner boundaries two similar and similarly situated ellipsoids; required the centre of gravity of the eighth part of it included between three principal planes. Let a, b, c be the semi-axes of the exterior ellipsoid, ra, rb, rc those of the inner ellipsoid, r being a quantity less than unity.

If a, b, c be the semi-axes of an ellipsoid, the volume of the eighth part is $\frac{1}{6}\pi abc$, and the co-ordinates of its centre of gravity are $\frac{3}{8}a$, $\frac{3}{8}b$, and $\frac{3}{8}c$ (see Art. 129). Hence

$$\tfrac{3}{8}a \,.\, \tfrac{1}{6}\pi abc = \tfrac{3}{8}ra \,.\, \tfrac{1}{6}\pi r^3 abc + \bar{x}\,(\tfrac{1}{6}\pi abc - \tfrac{1}{6}\pi r^3 abc);$$

therefore
$$\bar{x} = \frac{\frac{3}{8}a\,(1 - r^4)}{1 - r^3} = \tfrac{3}{8}a \,.\, \frac{1 + r + r^2 + r^3}{1 + r + r^2}.$$

If we suppose the shell indefinitely thin, we must put $r = 1$, and then $\bar{x} = \frac{1}{2}a$. Similar results may be found for $\bar{y}$ and $\bar{z}$.

(4) An ellipsoid is composed of an infinite number of indefinitely thin shells; each shell has for its outer and inner boundaries two similar and similarly situated ellipsoids; the density of each shell is constant, but the density varies from shell to shell according to a given law; determine the centre of gravity of the eighth part of the ellipsoid included between three principal planes.

Let x, y, z represent the three semi-axes of an ellipsoid; then the volume of the ellipsoid is $\frac{4\pi}{3}xyz$. Suppose that $y = mx$ and $z = nx$, where m and n are constants, then the volume becomes $\frac{4\pi mn}{3}x^3$, and if there be a *similar* ellipsoid having $x + \Delta x$ for the semi-axis corresponding to the semi-axis x of the first ellipsoid, the volume of the second ellipsoid will be $\frac{4\pi mn}{3}(x + \Delta x)^3$. Hence the volume of a shell bounded by two similar and similarly situated ellipsoids may be denoted by $\frac{4\pi mn}{3}\{(x + \Delta x)^3 - x^3\}$, and therefore by $4\pi mnx^2\Delta x$ when the thickness is indefinitely diminished. Let $\phi(x)$ denote the density of the shell, then its mass is $4\pi mn\phi(x)x^2\Delta x$. Thus the mass of the eighth part of the shell is $\frac{\pi mn}{2}\phi(x)x^2\Delta x$. And the abscissa of the centre of gravity of the shell measured along the semi-axis x is $\frac{x}{2}$, by the preceding example. Thus for the abscissa $\bar{x}$ of the centre of gravity we have

$$\bar{x} = \frac{\int_0^a \frac{\pi mn}{2}\phi(x)\frac{x}{2}x^2dx}{\int_0^a \frac{\pi mn}{2}\phi(x)x^2dx}$$

$$= \frac{\frac{1}{2}\int_0^a x^3\phi(x)dx}{\int_0^a x^2\phi(x)dx},$$

where a is the semi-axis of the external surface corresponding to the semi-axis x. When $\phi(x)$ is given the integrations may be completed; and when $\bar{x}$ is known, the other co-ordinates of the centre of gravity may be inferred from symmetry.

(5) A chord of an ellipse cuts off a segment of constant area; determine the locus of the centre of gravity of the segment.

If a chord cuts off a segment of constant area from a *circle*, it is evident from the symmetry of the figure that the locus of the centre of gravity of the segment is a *concentric circle*. Now if the circle be projected orthogonally upon a plane inclined to the plane of the circle the circle projects into an ellipse; and the segments of the circle of constant area project into segments of the ellipse of constant area; also the *concentric circle* projects into a second ellipse similar to the first ellipse and similarly situated. This second ellipse is the required locus, as is more fully shewn in a subsequent Article (139).

The following problem could in like manner be reduced to the corresponding problem for a sphere, by Art. 140.

(6) A plane cuts off from an ellipsoid a segment of constant volume; determine the locus of the centre of gravity of the segment.

Let the cutting plane have any position; and refer the ellipsoid to conjugate semi-diameters as axes; let the plane of (y, z) be parallel to the position of the cutting plane, and suppose the equation to the ellipsoid to be

$$\frac{x^2}{a'^2}+\frac{y^2}{b'^2}+\frac{z^2}{c'^2}=1.$$

Now suppose the segment cut off by the plane to be divided into an indefinitely large number of indefinitely thin slices by planes parallel to the plane of (y, z). By the properties of the ellipsoid these slices will be bounded by ellipses which have their centres on the axis of x; and thus we see that the centre of gravity of the segment cut off will be on the axis of x. Consider one of the slices bounded by planes which have for their abscissæ x and $x+\Delta x$ respectively; then it will be found that the volume of the slice is ultimately

$$\pi b'c'\left(1-\frac{x^2}{a'^2}\right)\sin\omega\sin\alpha\Delta x,$$

where ω is the angle between the axes of y and z, and α is the angle which the axis of x makes with the plane of (y, z). Suppose V to denote the constant volume, and $\lambda a'$ the abscissa of the plane cutting off the segment; then

$$V = \pi b'c' \sin\omega \sin\alpha \int_{\lambda a'}^{a'} \left(1 - \frac{x^2}{a'^2}\right) dx$$

$$= \pi a'b'c' \sin\omega \sin\alpha \{1 - \lambda - \frac{1}{3}(1 - \lambda^3)\}.$$

Now by the properties of the ellipsoid

$$\pi a'b'c' \sin\omega \sin\alpha = \pi abc,$$

where a, b, c are the semi-axes of the ellipsoid; thus

$$V = \pi abc \{1 - \lambda - \frac{1}{3}(1 - \lambda^3)\} \ldots\ldots\ldots\ldots\ldots (1).$$

And, if $\bar{x}$ be the abscissa of the centre of gravity of the segment cut off,

$$\bar{x} = \frac{\pi b'c' \sin\omega \sin\alpha \int_{\lambda a'}^{a'} x\left(1 - \frac{x^2}{a'^2}\right) dx}{V}$$

$$= \frac{\pi a'^2 b'c' \sin\omega \sin\alpha}{V} \left\{\frac{1}{2}(1 - \lambda^2) - \frac{1}{4}(1 - \lambda^4)\right\}$$

$$= \frac{\pi abc}{V} \left\{\frac{1}{2}(1 - \lambda^2) - \frac{1}{4}(1 - \lambda^4)\right\} a' \ldots\ldots\ldots (2).$$

Now (1) gives a constant value for λ, and then (2) shews that $\bar{x}$ *bears a constant ratio to* a'.

Thus the locus of the centre of gravity of segments of an ellipsoid of constant volume is an ellipsoid similar to the original ellipsoid and similarly situated.

(7) Water is gently poured into a vessel of any form: shew that, when so much water has been poured in that the centre of gravity of the vessel and water is in the lowest possible position, it will be in the surface of the water, provided that the centre of gravity of the empty vessel is higher than the lowest part of the water.

Suppose that when any quantity of water has been poured in, the centre of gravity of the vessel and the water is above the surface of the water. Let a small additional quantity of water be poured in; then the centre of gravity of the whole system is at some point between its former position and the

centre of gravity of the additional water: that is, the centre of gravity of the system has *descended* slightly. In this manner we shew that the centre of gravity of the system continually descends until it reaches the ascending surface of the water; and after that it continually ascends.

(8) O is the centre of a regular polyhedron, S is any point within it; from S perpendiculars are drawn on the faces, and equal particles are placed at the feet of these perpendiculars: shew that the centre of gravity of the system of particles is on the straight line OS, at the distance $\frac{2}{3}OS$ from O.

Take S for the origin of three rectangular axes, and let OS produced through S be the direction of the axis of x. Let $OS = c$.

Let p denote the length of the perpendicular from O on one of the faces of the polyhedron; let θ, ϕ, ψ be the angles which it makes with the axes of x, y, z respectively. Then the length of the perpendicular from S on the face is $p - c\cos\theta$; and therefore the co-ordinates of the foot of the perpendicular from S on the face will be respectively,

$$(p - c\cos\theta)\cos\theta, \quad (p - c\cos\theta)\cos\phi, \quad (p - c\cos\theta)\cos\psi.$$

Hence if n be the number of faces of the polyhedron, we have for the co-ordinates of the required centre of gravity

$$\bar{x} = \frac{\Sigma(p - c\cos\theta)\cos\theta}{n},$$

$$\bar{y} = \frac{\Sigma(p - c\cos\theta)\cos\phi}{n},$$

$$\bar{z} = \frac{\Sigma(p - c\cos\theta)\cos\psi}{n},$$

where the summation is to be taken with respect to the perpendiculars on all the faces.

Now it is obvious that

$$\Sigma p\cos\theta = 0, \quad \Sigma p\cos\phi = 0, \quad \Sigma p\cos\psi = 0,$$

because the centre of gravity of a system of equal particles

placed at the feet of perpendiculars from O on the faces would by symmetry coincide with O.

Moreover it follows from the principles explained in *Spherical Trigonometry*, Arts. 177 and 183, that

$$\Sigma \cos^2\theta = \frac{n}{3}, \quad \Sigma \cos\theta \cos\phi = 0, \quad \Sigma \cos\theta \cos\psi = 0.$$

Therefore $$\bar{x} = -\frac{c}{3}, \quad \bar{y} = 0, \quad \bar{z} = 0.$$

Thus the required result is established.

138. The following propositions are often useful.

If G be the centre of gravity of masses m_1, m_2, m_3 placed at the corners A, B, C of a triangle, the triangles BGC, CGA, AGB will be proportional to m_1, m_2, m_3.

For the total moment round the side BC in any position of the triangle is the same as if the three masses were collected at G; but m_2 and m_3 give no moment round BC, therefore the moment of $m_1 + m_2 + m_3$ at G is equal to the moment of m_1 at A. But the perpendiculars from G and A on BC are as the areas GBC, ABC. Thus we have

$$\frac{m_1 + m_2 + m_3}{ABC} = \frac{m_1}{BGC} = \frac{m_2}{CGA} = \frac{m_3}{AGB}.$$

This proposition affords the readiest mode of determining the ratios of m_1, m_2, m_3 when the position of G in the triangle ABC is given.

If G be the centre of gravity of masses m_1, m_2, m_3, m_4 placed at the corners A, B, C, D of a tetrahedron, the tetrahedrons $GBCD$, $GCDA$, $GDAB$, $GABC$ are proportional to m_1, m_2, m_3, m_4.

For the distance of G from the plane BCD is

$$\frac{m_1}{m_1 + m_2 + m_3 + m_4}$$

of the distance of A, and these distances are as the volumes $GBCD$, $ABCD$. Therefore we have

$$\frac{m_1+m_2+m_3+m_4}{ABCD}=\frac{m_1}{GBCD}=\frac{m_2}{GCDA}=\frac{m_3}{GDAB}=\frac{m_4}{GABC}.$$

139. When a plane area is orthogonally projected, the projection of its centre of gravity coincides with the centre of gravity of the projection.

Take the axis of x in the original area parallel and the axes of y perpendicular to the intersection of the two planes, and let the cosine of the angle between the planes be denoted by c. Let the new axes of ξ and η be the projections of the old axes of x and y; they will therefore be at right angles. Comparing any point of the original area with its projection, we shall have

$$\xi=x, \quad \eta=cy, \quad d\xi d\eta=c\,dx\,dy;$$

hence

$$\bar{\xi}=\frac{\int\xi d\xi d\eta}{\int d\xi d\eta}=\frac{c\int x\,dx\,dy}{c\int dx\,dy}=\bar{x},$$

$$\bar{\eta}=\frac{\int\eta d\xi d\eta}{\int d\xi d\eta}=\frac{c^2\int y\,dx\,dy}{c\int dx\,dy}=c\bar{y};$$

therefore the point $(\bar{\xi}, \bar{\eta})$ is the projection of the point $(\bar{x}, \bar{y})$.

More generally, if two areas are so related that (the axes being rectangular) $\xi=ax$, $\eta=by$, where a and b are constants, we shall have $\bar{\xi}=a\bar{x}$, $\bar{\eta}=b\bar{y}$.

For

$$\bar{\xi}=\frac{\int\xi d\xi d\eta}{\int d\xi d\eta}=\frac{a^2b\int x\,dx\,dy}{ab\int dx\,dy}=a\bar{x},$$

$$\bar{\eta}=\frac{\int\eta d\xi d\eta}{\int d\xi d\eta}=\frac{ab^2\int y\,dx\,dy}{ab\int dx\,dy}=b\bar{y}.$$

If two solid figures are so related that (the axes being rectangular) $\xi=ax$, $\eta=by$, $\zeta=cz$, where ξ, η, ζ are the coordinates of the point in one figure which corresponds to the point (x, y, z) in the other, the centres of gravity of the two figures will be corresponding points.

For

$$\bar{\xi}=\frac{\int\xi d\xi d\eta d\zeta}{\int d\xi d\eta d\zeta}=\frac{a^2bc\int x\,dx\,dy\,dz}{abc\int dx\,dy\,dz}=a\bar{x}.$$

Similarly we can shew that $\bar{\eta} = b\bar{y}$, $\bar{\zeta} = c\bar{z}$.

140. If two plane areas are so related to each other that, when one is referred to rectangular axes of x and y, and the other to oblique axes of ξ and η, the ratios of ξ to x and of η to y for corresponding points have constant values a, b, then the centres of gravity of the two areas will be corresponding points.

For we have $\xi = ax$, $\eta = by$, and by Art. 115

$$\bar{\xi} = \frac{\int \xi d\xi d\eta}{\int d\xi d\eta} = \frac{a^2 b \int x dx dy}{ab \int dx dy} = a\bar{x}.$$

Similarly we have $\bar{\eta} = b\bar{y}$.

If two solid figures are so related to each other that when one is referred to rectangular axes of x, y, z, and the other to oblique axes of ξ, η, ζ, the ratios of ξ to x of η to y and of ζ to z for corresponding points have constant values a, b, c, then the centres of gravity of the two volumes are corresponding points.

For we have $\xi = ax$, $\eta = by$, $\zeta = cz$, and by Art. 128

$$\bar{\xi} = \frac{\int \xi d\xi d\eta d\zeta}{\int d\xi d\eta d\zeta} = \frac{a^2 bc \int x dx dy dz}{abc \int dx dy dz} = a\bar{x}.$$

Similarly we have $\bar{\eta} = b\bar{y}$, $\bar{\zeta} = c\bar{z}$.

From the first of these propositions it follows, by comparison with a semi-circle, that the centre of gravity of a semi-ellipse cut off by any diameter lies upon the conjugate semi-axis at the distance $\frac{2}{\pi}$ of its length from the centre; and from the second it follows, by comparison with a hemisphere, that the centre of gravity of a semi-ellipsoid cut off by any plane lies on the conjugate semi-axis at the distance of $\frac{3}{8}$ of its length from the centre.

140 A. Theorem of Leibnitz. If n equal particles are placed at points A_1, A_2, ... A_n, and G denote their centre of gravity, the resultant of forces represented by lines OA_1,

$OA_2, \ldots OA_n$ drawn from any origin O is a force represented by $n \,.\, OG$.

For if A_1 be (x_1, y_1, z_1), &c., we have $\bar{x} = \frac{1}{n}(x_1 + x_2 \ldots + x_n)$; but the components of the force OA_1 are x_1, y_1, z_1 and the total x-component of the resultant is $x_1 + x_2 \ldots + x_n = X$ (suppose). Thus we have $X = n\bar{x}$, $Y = n\bar{y}$, $Z = n\bar{z}$, which proves the proposition.

140 B. If O and G coincide, the resultant force vanishes, and conversely if the forces are in equilibrium O and G must coincide, that is to say:—*If forces acting at a point and represented by lines radiating from this point are in equilibrium, the point must be the centre of gravity of equal masses supposed to be placed at the extremities of the radiating lines.*

140 C. More generally, if masses $m_1, m_2, \ldots m_n$ are placed at the points, and G is their centre of gravity, we shall have

$$(m_1 + m_2 \ldots + m_n)\,\bar{x} = m_1 x_1 + m_2 x_2 \ldots + m_n x_n,$$

and if forces $m_1 \,.\, OA_1$, $m_2 \,.\, OA_2 \ldots m_n \,.\, OA_n$ act along OA_1, $OA_2 \ldots OA_n$ we have

$$X = m_1 x_1 + m_2 x_2 \ldots + m_n x_n.$$

Thus we have $X = \bar{x}\Sigma m$, $Y = \bar{y}\Sigma m$, $Z = \bar{z}\Sigma m$,

and the resultant force is represented by $OG \,.\, \Sigma m$.

By supposing the lines $OA_1, OA_2, \ldots OA_n$ to be equal, we obtain the following proposition. *If forces act along equal lines radiating from a point, the condition of their equilibrium is that this point be the centre of gravity of masses proportional to the forces, placed at the extremities of the radiating lines.* This latter proposition combined with the latter part of Art. 138 gives another proof of Prop. X. Chap. II.

140 D. A *vector* is a directed quantity of any kind that follows the parallelogram law of composition. The velocity of a particle is the simplest example. Forces applied to the same particle, and couples applied to the same rigid body are vectors.

The resultant of two vectors is called their *sum*. Hence we obtain the definition of their difference. If α and β are two vectors, their difference $\alpha - \beta$ is a vector γ such that α is the resultant of β and γ.

A vector can be represented by a straight line having the same direction as the vector and proportional to it in magnitude. As there are two directions along any straight line, some means must be employed to indicate which of the two is to be taken. In a diagram the indication may be given by an arrow-head marked on the line. In naming the line by the letters at its two ends the direction is indicated by the order of the letters. Thus the vector AB is *minus* the vector BA.

If A, B, C are any three points, the sum of the vectors AB and BC is the vector AC.

The notation $\overline{AB}$ is used to signify "the vector AB". Thus we have always

$$\overline{AB} + \overline{BC} = \overline{AC}, \text{ and } \overline{AC} - \overline{AB} = \overline{BC}.$$

The order of the terms is indifferent; for example we have

$$\overline{AC} - \overline{AB} = \overline{AC} + \overline{BA} = \overline{BA} + \overline{AC} = \overline{BC}.$$

A vector may be multiplied by a numerical quantity. This operation only affects its length, not its direction, unless the multiplier be negative, in which case the direction is reversed.

The method of vectors may be used with advantage in certain problems relating to centres of gravity, as in the following example.

140 E. Let a mass m_1 be placed at a point A_1, and a mass m_2 at a point A_2. Find their centre of gravity B by dividing the line A_1A_2 so that $m_1 . \overline{A_1B} = m_2 . \overline{BA_2}$, and from any point O draw $\overline{OA_1}$, $\overline{OA_2}$, $\overline{OB}$. Then we have the vector equations

$$m_1 . \overline{OA_1} + m_2 . \overline{OA_2} = m_1(\overline{OB} + \overline{BA_1}) + m_2(\overline{OB} + \overline{BA_2})$$
$$= (m_1 + m_2)\overline{OB} + m_1 . \overline{BA_1} + m_2 . \overline{BA_2}.$$

But $m_1 . \overline{BA_1} + m_2 . \overline{BA_2} = - m_1 . \overline{A_1B} + m_2 . \overline{BA_2} = 0,$
by construction. Therefore

$$m_1 . \overline{OA_1} + m_2 . \overline{OA_2} = (m_1 + m_2) . \overline{OB}.$$

Let there be a third mass m_3 at A_3, and (as in Art. 66) let BA_3 be divided at C so that

$$(m_1 + m_2) . \overline{BC} = m_3 . \overline{CA_3}.$$

Then by employing the foregoing result, we have

$$(m_1 + m_2) . \overline{OB} + m_3 . \overline{OA_3} = (m_1 + m_2 + m_3) . \overline{OC};$$

that is,

$$m_1 . \overline{OA_1} + m_2 . \overline{OA_2} + m_3 . \overline{OA_3} = (m_1 + m_2 + m_3) . \overline{OC}.$$

The process may be extended to include any number of points, thus we have the general result

$$\Sigma (m . \overline{OA}) = \overline{OG} . \Sigma m,$$

G denoting the centre of gravity of the masses at the points A, and O being any point whatever.

140 F. If there are n masses and they are all equal, the formula becomes

$$\Sigma . \overline{OA} = n . \overline{OG},$$

which expresses the theorem of Leibnitz (Art. 140 A).

A vector (when not limited to a given plane) requires three numerical quantities to specify it. These may be its magnitude and the two angles which specify its direction, or they may be the numerical values of its components parallel to three rectangular axes of reference. The equality of two vectors, which is expressed by such equations as the above, accordingly implies three equations of the ordinary kind. Thus if O in the above investigation is the origin, and x_1, y_1, z_1 are the co-ordinates of A_1, the last of the above equations is equivalent to the three simultaneous equations

$$\Sigma x = n\bar{x}, \quad \Sigma y = n\bar{y}, \quad \Sigma z = n\bar{z}.$$

141. The volume (V) of a frustum of a cylinder intercepted between two planes, one of which is perpendicular to

the axis of the cylinder, is obviously given by the equation

$$V = \iint z\,dx\,dy,$$

where the plane of (x, y) is the plane perpendicular to the axis, and z is the ordinate of a point in the other plane.

Let ϕ denote the angle between the two planes, and let their line of intersection be taken as axis of x. Then we shall have

$$z = y \tan\phi, \quad V = \tan\phi \iint y\,dx\,dy = \tan\phi \,.\, \bar{y} \iint dx\,dy,$$

that is, the volume is equal to the area of the normal section multiplied by the height of the frustum at the centre of gravity of the normal section. The line which measures this height also passes through the centre of gravity of the oblique section; for we have virtually proved in Art. 139 that the centres of gravity of all plane sections of a cylinder (in the most general sense of this word) lie in the same straight line parallel to the generating lines of the cylinder.

A frustum of a cylinder bounded by two oblique sections may be regarded as the difference of two frusta bounded at one end by the same normal section. If A denote the area of this common base and $\bar{z}_1$, $\bar{z}_2$ their heights measured along the line which contains the centres of gravity of all sections, their volumes will be $A\bar{z}_1$, $A\bar{z}_2$, and the difference of these is $A(\bar{z}_1 \sim \bar{z}_2)$, that is, the volume of any frustum of a cylinder with plane ends is equal to the area of a normal section multiplied by the distance between the centres of gravity of its ends.

142. The following properties of the centre of gravity have important physical applications.

Let $\bar{x}, \bar{y}, \bar{z}$ be the co-ordinates of the centre of gravity of a system of particles referred to any origin and rectangular axes,

x_1, y_1, z_1 the co-ordinates of the first particle, m_1 its mass,

ξ_1, η_1, ζ_1 the co-ordinates of this particle referred to the centre of gravity as origin, with similar notation for the other particles.

Then we have, by hypothesis,

$$x_1 = \bar{x} + \xi_1, \quad y_1 = \bar{y} + \eta_1, \quad z_1 = \bar{z} + \zeta_1;$$

$$m_1 x_1^2 + m_2 x_2^2 + \ldots = (m_1 + m_2 + \ldots)\,\bar{x}^2 + m_1 \xi_1^2 + m_2 \xi_2^2 + \ldots + 2\bar{x}\,(m_1 \xi_1 + m_2 \xi_2 + \ldots).$$

But $m_1 \xi_1 + m_2 \xi_2 + \ldots = 0$, since ξ_1, $\xi_2 \ldots$ are measured from the centre of gravity. Therefore we have

$$\Sigma m x^2 = \bar{x}^2 \Sigma m + \Sigma m \xi^2,$$

$$\Sigma m y^2 = \bar{y}^2 \Sigma m + \Sigma m \eta^2,$$

$$\Sigma m z^2 = \bar{z}^2 \Sigma m + \Sigma m \zeta^2.$$

By adding the first two of these equations, we have

$$\Sigma m\,(x^2 + y^2) = (\bar{x}^2 + \bar{y}^2)\,\Sigma m + \Sigma m\,(\xi^2 + \eta^2),$$

where $x^2 + y^2$ is the square of the distance of the particle m from the axis of z, $\xi^2 + \eta^2$ is the square of its distance from the axis of ζ, and $\bar{x}^2 + \bar{y}^2$ is the square of the distance between these two parallel axes.

By adding all three equations we obtain a result which may be written

$$\Sigma m r^2 = h^2 \Sigma m + \Sigma m \rho^2,$$

r denoting the distance of the particle m from the first origin, ρ its distance from the centre of gravity, and h the distance between the first origin and the centre of gravity.

The student may prove by similar reasoning that

$$\Sigma m x y = \bar{x}\bar{y} \Sigma m + \Sigma m \xi \eta.$$

143. *Through the centre of gravity of each face of a tetrahedron a force acts at right angles to the face, and proportional to the area of the face: if the forces all act inwards or all act outwards they will be in equilibrium.*

Let A, B, C, D denote the angular points of the tetrahedron. The force acting on the face ABC, at its centre of gravity, may be replaced by three equal forces acting at right angles to the face at the points A, B, C respectively. Similar substitutions may be made for the other forces. Thus we have, acting at the point A, three forces respectively at right angles to the three faces which meet at A and proportional to the areas of those faces; and, by what has been shewn in the Propositions at the end of Chapter v., these three forces are equivalent to a single force acting at A in the direction perpendicular to the face BCD, and proportional to the area of that face. Hence, by Proposition i. at the end of Chapter v., the proposed system of forces will be in equilibrium.

The preceding result may now be extended to the following proposition: *Through the centre of gravity of each face of a polyhedron a force acts at right angles to the face, and proportional to the area of the face: if the forces all act inwards or all act outwards they will be in equilibrium.*

For each face of the polyhedron may be divided into triangles; and the force acting at the centre of gravity of the face may be replaced by forces acting respectively at the centres of gravity of the triangles, and proportional to the areas of the triangles. Then the polyhedron may be supposed to be made up of tetrahedrons which have a common vertex, and two equal and opposite forces may be supposed applied at every common face, acting through the centre of gravity of the face at right angles to the face and proportional to the area of the face. Hence the required result follows from the former part of this Article in the manner already exemplified in Proposition i. at the end of Chapter iv.

The preceding general result was first brought under the notice of the present writer by the late Bishop Mackenzie; it was given in an examination paper in Gonville and Caius College in 1849, probably by himself. The method by which he demonstrated it will be found interesting and instructive by the student who is acquainted with Hydrostatics. Imagine a fluid in equilibrium acted on by no forces; then the pressure will be constant throughout the mass. Sup-

pose a portion of the fluid in the form of a polyhedron to become solid; then the equilibrium will not be disturbed. The forces acting on the faces of the polyhedron will be respectively at right angles to the faces and proportional to the areas of the faces, and will act through the centres of gravity of the faces. Hence the required result follows.

The proposition may have been enunciated previously; however a very eminent mathematician stated at the meeting of the British Association at Cheltenham in 1856, that he had been unable to find it in print.

By means of Art. 50 we can deduce the following proposition respecting couples: *A system of couples represented in position and magnitude by the faces of a polyhedron will be in equilibrium, supposing the axes of the couples all to be directed inwards or all outwards.* This is given by Möbius; *Lehrbuch der Statik*, Vol. I. page 87.

Guldinus's Properties.

144. *If any plane figure revolve about an axis lying in its plane, the content of the solid generated by this figure in revolving through any angle is equal to a prism, of which the base is the revolving figure and height the length of the path described by the centre of gravity of the area of the plane figure.*

The axis of revolution in this and the following proposition is supposed not to cut the generating curve.

Let the axis of revolution be the axis of x, and the plane of the revolving figure in its initial position the plane of (x, y); let β be the angle through which the figure revolves.

The elementary area $\Delta x\, \Delta y$ of the plane figure in revolving through an angle $\Delta\theta$ generates the elementary solid whose volume is $y\Delta\theta\, \Delta x\, \Delta y$; therefore the whole solid

$$= \iiint_0^\beta y\, dx\, dy\, d\theta = \beta \iint y\, dx\, dy.$$

The limits of x and y depend on the nature of the curve. But if $\overline{y}$ be the ordinate to the centre of gravity of the plane figure, then, by Art. 118,

$$\overline{y} = \frac{\iint y\,dx\,dy}{\iint dx\,dy},$$

the limits being the same as before.

Therefore the whole solid $= \beta\iint y\,dx\,dy = \overline{y}\beta\iint dx\,dy =$ the arc described by the centre of gravity multiplied by the area of the figure.

If any plane figure revolve about an axis lying in its plane, the surface of the solid generated is equal in area to the rectangle, of which the sides are the length of the perimeter of the generating figure and the length of the path of the centre of gravity of the perimeter.

The surface generated by the arc Δs of the figure revolving through an angle $\Delta\theta$ is $y\Delta\theta\,\Delta s$; therefore the whole surface

$$= \iint_0^\beta y\,ds\,d\theta = \beta\int y\,ds.$$

The limits depend on the nature of the curve. But if $\overline{y}$ be the ordinate to the centre of gravity of the perimeter,

$$\overline{y} = \frac{\int y\,ds}{\int ds},$$

the limits being the same as before.

Therefore the whole surface $= \overline{y}\beta\int ds =$ the arc described by the centre of gravity, multiplied by the length of the perimeter.

Ex. 1. *To find the solid content and the surface of the ring formed by the revolution of a circle round a straight line in its own plane which it does not meet.*

Let the distance of the centre of the circle from the axis of revolution be a; let b be the radius of the circle; then the length of the path of the centre of gravity of the area of the figure is $2\pi a$, and the area of the figure is πb^2;

therefore the content of the solid $= 2\pi^2 ab^2$.

Also the length of the path of the centre of gravity of the perimeter is $2\pi a$, and the length of the perimeter is $2\pi b$;

therefore the surface of the solid $= 4\pi^2 ab$.

Ex. 2. *To find the centre of gravity of the area and also of the arc of a semicircle.*

A semicircle by revolving about its diameter generates a sphere; the content of the sphere is $\frac{4}{3}\pi a^3$, and the surface $4\pi a^2$, the radius being a; the area of the semicircle is $\frac{1}{2}\pi a^2$, and the perimeter πa; therefore, the distance of the centre of gravity of the *area* from the diameter

$$= \frac{\text{content of sphere}}{2\pi \,.\, \text{area of semicircle}} = \frac{4a}{3\pi};$$

the distance of the centre of gravity of the arc from the diameter

$$= \frac{\text{surface of sphere}}{2\pi \,.\, \text{arc of semicircle}} = \frac{2a}{\pi}.$$

Ex. 3. *To find the surface and the solid content of the solid formed by the revolution of a cycloid round the tangent at its vertex.*

In Art. 133 we have found $\frac{2a}{3}$ for the distance of the centre of gravity of the arc of a cycloid from its vertex; and the whole length of the arc is $8a$. Therefore the surface of the solid generated is

$$2\pi \times \frac{2a}{3} \times 8a; \text{ that is } \frac{32}{3}\pi a^2.$$

And in Art. 113 we have found that the distance of the centre of gravity of the area included between the cycloid and its base from the vertex is $\frac{7}{6}a$; and the area so included is $3\pi a^2$. Hence the area of the portion which in the present case revolves round the tangent is $4\pi a^2 - 3\pi a^2$, that is πa

And the centre of gravity of this area may be shewn to be at a distance $\frac{a}{2}$ from the vertex. (See Ex. (2) of Art. 109.) Therefore the solid content of the figure generated is $2\pi \frac{a}{2} \pi a^2$, that is $\pi^2 a^3$.

144 A. The restriction stated in Art. 144—that the axis of revolution is supposed not to cut the generating curve—can be removed if we pay proper attention to sign. Let one face of the area be regarded as positive, and a normal projecting from this face be regarded as indicating the positive direction. Then volume is to be regarded as positive or negative according as it is generated by motion in the positive or in the negative direction.

Any small motion of a plane area can be resolved into the following steps:—

1. A motion of translation in which one particular point selected at pleasure moves into its new position.

2. A rotation round an axis (through the point) in the plane of the area until the area comes into its new plane.

3. Rotation in this plane (about an axis perpendicular to the plane through the selected point) until the area comes into its new position.

In the first of these movements the area generates a prism, of which the area is the base, and the displacement of the selected point resolved normally to the plane is the height.

In the second, the volume generated is (by Art. 144) equal to a prism, of which the area is the base, and the path of the centre of gravity the height. This volume vanishes if the centre of gravity itself is the selected point.

In the third, no volume is generated.

Hence we have the following conclusions.

No volume (in algebraic sense) is generated by a plane area in any motion in which its centre of gravity remains at rest.

The volume (in algebraic sense) *generated in any small motion of a plane area is equal to the area multiplied by the displacement of its centre of gravity resolved in a direction normal to the area.*

The volume generated in any motion of the area is equal to the area multiplied by the integral $\int \cos\theta \, . \, ds$, where ds denotes an element of the path of the centre of gravity, and θ the angle which this element makes with a normal drawn on the positive side of the area.

In like manner it can be shewn that the area generated by the motion of a straight line *in plano*, is equal to the length of the line multiplied by the normal movement of its middle point.

EXAMPLES.

1. Find the centre of gravity of five equal heavy particles placed at five of the angular points of a regular hexagon.

2. Five pieces of a uniform chain are hung at equidistant points along a rigid rod without weight, and their lower ends are in a straight line passing through one end of the rod; find the centre of gravity of the system.

3. A plane quadrilateral $ABCD$ is bisected by the diagonal AC, and the other diagonal divides AC into two parts in the ratio of p to q; shew that the centre of gravity of the quadrilateral lies in AC and divides it into two parts in the ratio of $2p + q$ to $p + 2q$.

4. From the fact that any system of heavy particles has one centre of gravity and only one, deduce the property that the straight lines joining the middle points of the opposite sides of any quadrilateral figure bisect each other.

5. A pyramid stands on a square base: given the co-ordinates of the vertex, and the co-ordinates of two opposite corners of the base, determine the co-ordinates of the centre of gravity of the pyramid.

6. ABC is a triangle; D, E, F are the middle points of its sides; shew that the centre of gravity of the *sides* of ABC coincides with the centre of the circle inscribed in DEF.

7. A piece of wire is formed into a triangle; find the distance of the centre of gravity from each of the sides, and shew that if x, y, z be the three distances, and r the radius of the inscribed circle, then

$$4xyz - r^2(x+y+z) - r^3 = 0.$$

8. If the centre of gravity of a four-sided figure coincide with one of its angular points, shew that the distances of this point and the opposite angular point from the straight line joining the other two angular points are as 1 to 2.

9. Shew that the common centre of gravity of a right-angled isosceles triangle, and the squares described on the two equal sides, is at a distance $= \frac{\sqrt{2}}{15} a$ from the point in which those sides meet, a being the length of one of them.

10. Prove the following construction for the centre of gravity of any quadrilateral. Let E be the intersection of the diagonals, and F the middle point of the straight line which joins their middle points; draw the straight line EF and produce it to G, making $FG = \frac{1}{3}EF$; then G shall be the centre of gravity required.

11. A triangle ABC is successively suspended from the angles A and B, and the two positions of any side are at right angles to each other; shew that

$$5c^2 = a^2 + b^2.$$

12. A right-angled triangular lamina ABC is suspended from a point D in its hypothenuse AB; prove that in the position of equilibrium AB will be horizontal if

$$AD : DB :: AB^2 + AC^2 : AB^2 + BC^2.$$

13. A given isosceles triangle is inscribed in a circle; find the centre of gravity of the remaining area of the circle.

14. If three uniform rods be rigidly united so as to form half of a regular hexagon, prove that if suspended from one of the angles, one of the rods will be horizontal.

15. If ABC be an isosceles triangle having a right angle at C, and D, E be the middle points of AC, AB respectively, prove that a perpendicular from E upon BD will pass through the centre of gravity of the triangle BDC.

16. $ABCD$ is any plane quadrilateral figure, and a, b, c, d are respectively the centres of gravity of the triangles BCD, CDA, DAB, ABC; shew that the quadrilateral $abcd$ is similar to $ABCD$.

17. A, B, C, D, E, F are six equal particles at the angles of *any* plane hexagon, and a, b, c, d, e, f are the centres of gravity respectively of ABC, BCD, CDE, DEF, EFA, and FAB. Shew that the opposite sides and angles of the hexagon $abcdef$ are equal, and that the straight lines joining opposite angles pass through one point, which is the centre of gravity of the particles A, B, C, D, E, F.

18. A straight line ED cuts off $\frac{1}{n}$ th part of the right-angled triangle ABC of which A is the right angle. $AB = a$, $AC = b$. Shew that the centre of gravity of $CEDB$ describes the curve whose equation is

$$\frac{ab}{n} = \{3(n-1)\,y - nb\}\{3(n-1)x - na\}.$$

19. The distance of the centre of gravity of any number of sides AB, BC, CD.....KL of a regular polygon from the centre of the inscribed circle

$$= \frac{AL \times \text{radius}}{AB + BC + CD + \ldots\ldots + KL}.$$

20. A frustum is cut from a right cone by a plane bisecting the axis and parallel to the base; shew that it will rest with its slant side on a horizontal table if the height of the cone bear to the diameter of its base a greater ratio than $\sqrt{7}$ to $\sqrt{17}$.

21. If particles of unequal weights be placed at the angular points of a triangular pyramid, and G_1 be their common centre of gravity; G_2, G_3,... their common centres of gravity for every possible arrangement of the particles; shew that the centre of gravity of equal particles placed at G_1, G_2,... is the centre of gravity of the pyramid.

22. If a cone have its base united concentrically to the base of a hemisphere of equal radius, find the height of the cone that the solid may rest on a horizontal table on any point of its spherical surface. *Result.* $r\sqrt{3}$.

23. If any polygon circumscribe a circle, the centre of gravity of the *area* of the polygon, the centre of gravity of the *perimeter* of the polygon, and the centre of the circle, are in the same straight line; also the distance of the first point from the third is two-thirds of the distance of the second point from the third.

24. If any polyhedron circumscribe a sphere, the centre of gravity of the *volume* of the polyhedron, the centre of gravity of the *surface* of the polyhedron, and the centre of the sphere, are in the same straight line; also the distance of the first point from the third is three-fourths of the distance of the second point from the third.

25. From a right cone the diameter of whose base is equal to its altitude is cut a right cylinder the diameter of whose base is equal to its altitude, their axes being in the same straight line and the base of the cylinder lying in the base of the cone; from the remaining cone a similar cylinder is cut, and so on, indefinitely; shew that the distance of the centre of gravity of all the cylinders from the base of the cone is $\frac{3}{10}$ of the height of the cone, and that the distance of the centre of gravity of the remaining portion from the base of the cone is $\frac{17}{80}$ of the altitude of the cone.

26. A square is cut from an equilateral triangle, a side of the square coinciding with a side of the triangle; from the equilateral triangle which remains another square is cut, and so on, *ad infinitum:* find the centre of gravity of the sum of the squares.

27. Find the centre of gravity of the area contained between the curves $y^2 = ax$ and $y^2 = 2ax - x^2$, which is above the axis of x.

Results. $\bar{x} = a \cdot \dfrac{15\pi - 44}{15\pi - 40}$; $\bar{y} = \dfrac{a}{3\pi - 8}$.

28. Find the centre of gravity of the area enclosed by the curve $r = a(1 + \cos\theta)$. *Result.* $\bar{x} = \frac{5}{6}a$.

29. Find the centre of gravity of the area included by a loop of the curve $r = a\cos 2\theta$. *Result.* $\bar{x} = \dfrac{128a\sqrt{2}}{105\pi}$.

30. Find the centre of gravity of the area included by a loop of the curve $r = a\cos 3\theta$. *Result.* $\bar{x} = \dfrac{81\sqrt{3}a}{80\pi}$.

31. The locus of the centre of gravity of all equal segments cut off from a parabola is an equal parabola.

32. Find the centre of gravity of a segment of a circle.

33. Find the centre of gravity of the area included by the curves $y^2 = ax$ and $x^2 = by$.

Results. $\bar{x} = \frac{9}{20} a^{\frac{1}{3}} b^{\frac{2}{3}}$, $\bar{y} = \frac{9}{20} a^{\frac{2}{3}} b^{\frac{1}{3}}$.

34. Find the centre of gravity of a portion of an equilateral hyperbola bounded by the curve, the transverse axis, and a radius vector drawn from the centre.

Results. $$\frac{\bar{x}}{y'} = \frac{\bar{y}}{x' - a} = \frac{2}{3 \log (x' + y') - 3 \log a};$$

where x', y' are the co-ordinates of the point of intersection of the curve and the bounding radius vector.

35. Two equal circles of radius a are drawn, each passing through the centre of the other, and a third circle touches both, having one of their points of intersection for its centre; the distance of the centre of gravity of the smaller area included between the outer and inner circles from the common radius of the first two is

$$\frac{12 - 2\pi\sqrt{3}}{2\pi - 3\sqrt{3}}\, a.$$

36. The density of a triangle varies as the n^{th} power of the distance from the base; determine n when the centre of gravity of the triangle divides the straight line joining the vertex with the middle point of the base in the ratio of 3 to 1.

Result. $n = -\frac{1}{3}$.

37. Find the centre of gravity of the volume formed by the revolution round the axis of x of the area of the curve $y^4 - axy^2 + x^4 = 0$.

Result. $\bar{x} = \frac{3a\pi}{32}$.

38. Find the centre of gravity of the volume generated by the revolution of the area in Ex. 27 round the axis of y.

Result. $\bar{y} = \frac{5a}{2(15\pi - 44)}$.

39. Find the centre of gravity of a hemisphere when the density varies as the square of the distance from the centre.

Result. $\bar{x} = \frac{5a}{12}$.

40. Find the centre of gravity of the solid generated by a semiparabola bounded by the latus rectum revolving round the latus rectum.

Result. Distance from focus $= \frac{5}{3\frac{1}{2}}$ of latus rectum.

41. The solid included between the surfaces of a continuous hyperboloid and its conical asymptote is cut by two planes perpendicular to their common axis; find the position of the centre of gravity of that portion which lies between the planes.

Result. Midway between the planes.

42. A solid sector of a sphere hangs from a point in its circular rim with its axis horizontal, find its vertical angle.

Result. The cosine of the semivertical angle is $\frac{3}{5}$.

43. A solid is generated by the revolution of a semicircle about a straight line perpendicular to the diameter, and which does not meet the semicircle: find the distance of the centre of gravity from the plane generated by the diameter.

Result. $\frac{4r}{3\pi}$.

44. A is a point in the generating line of a right cylinder on a circular base, and B, C are two others in the generating line diametrically opposite. The cylinder is bisected by a plane ABC, and one of the semicylinders is cut by two planes at right angles to ABC, passing through AB and AC. Shew that if the solid ABC be placed with its convex side on a horizontal plane, the plane ABC will be inclined to the horizon at an angle $\tan^{-1}(\frac{3}{16}\pi)$, when there is equilibrium.

45. A solid cone is cut by two planes perpendicular to the same principal section, one through its axis, and the other parallel to a slant slide; find the limiting value of the cosine of the vertical angle of the cone, that the piece cut out may rest on its curved surface on a horizontal plane.

Result. $\frac{5}{9}$.

46. A quadrant of a circle revolves round one of its extreme radii through an angle of 30^0; find the centre of gravity of the solid traced out, the density being supposed to vary as the distance from the centre.

Results. $\bar{x} = \frac{3a}{5}$; $\bar{y} = \frac{3a}{5}(2-\sqrt{3})$; $\bar{z} = \frac{2a}{5}$. The axis of x is supposed to coincide with the initial position of the revolving radius.

47. A solid is formed by the revolution of the area of the curve $y^{2n-2} = ax^{2-n}$ round the axis of x; shew that the distance of the centre of gravity of any segment of this solid from the vertex bears to the height of the segment the ratio of 1 to n. The segment is supposed cut off by a plane perpendicular to the axis.

48. Find the centre of gravity of the *surface* of the solid $z^2 + y^2 = 2ax$, cut off by the plane $x = c$.

Result. $\bar{x} = \frac{(3c-a)(a+2c)^{\frac{3}{2}} + a^{\frac{5}{2}}}{5\{(a+2c)^{\frac{3}{2}} - a^{\frac{3}{2}}\}}$.

49. Apply Guldinus's theorem to find the volume of the frustum of a right cone in terms of its altitude and the radii of its ends.

Result. $\frac{h\pi}{3}(R^2 + Rr + r^2)$.

50. Find the surface and the volume of the solid formed by the revolution of a cycloid round its base.

Results. $\frac{64\pi a^2}{3}$; $5\pi^2 a^3$.

51. A segment of a circle revolves round its chord, which subtends an angle of 90^0 at the centre; find the surface and volume of the solid generated.

Results. $\frac{\pi a^2(4-\pi)}{\sqrt{2}}$; $\frac{a^3(10-3\pi)\pi}{6\sqrt{2}}$.

52. An ellipse whose excentricity is $\frac{4}{3\pi}$ revolves about any tangent line. Prove that the volume generated by one portion into which the ellipse is divided by its minor axis varies inversely as the volume generated by the other portion.

53. A plane area moves in such a manner as to be always normal to the curve along which its centre of gravity moves; prove that the volume generated is equal to the given area multiplied by the length of the path of the centre of gravity.

Hence find the volume of a cycloidal tube whose normal section is of constant area.

54. Extend Guldinus's theorem for finding the volume of a ring to the case in which the ring is formed by the revolution of a plane area about a straight line *parallel* to its plane.

A ring is formed by the revolution of the lemniscate (whose equation is $r^2 = a^2 \cos 2\theta$) about a straight line parallel to its plane situated in a plane drawn through its double point and perpendicular to its axis: shew that the volume of this ring is $\frac{\pi^2 a^3}{4\sqrt{2}}$.

CHAPTER IX.

"THE MECHANICAL POWERS."

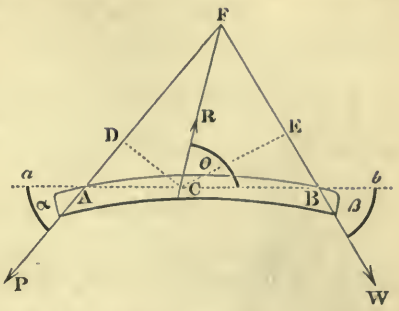
145. THE so-called "Mechanical Powers" or "Simple Machines" are certain arrangements which have been rendered familiar by the prominent place assigned to them in treatises on Mechanics from the earliest times. These we shall explain, and also a few combinations of them.

146. A *Lever* is an inflexible rod moveable only about a fixed axis, which is called the *fulcrum*. The portions of the lever into which the fulcrum divides it are called the *arms* of the lever: when the arms are in the same straight line, it is called a *straight lever*, and in other cases a *bent lever*.

Two forces act on the lever about the fulcrum, called the *power* and the *weight:* the power is the force applied by the hand, or other means, to sustain or overcome the other force, or the weight. There are three species of levers: in the first the fulcrum is between the power and the weight; in the second the weight acts between the fulcrum and the power; and in the third the power acts between the fulcrum and the weight.

147. *To find the conditions of equilibrium of two forces acting in the same plane on a lever.*

Let the plane of the paper be the plane in which the forces act, and also be perpendicular to the axis, of which C is the projection, and about which the lever can move; A, B the points of application of the forces P, W; α, β the angles which the directions of the forces make with any straight line aCb drawn through C on the paper. Let R be the pressure on the fulcrum, and θ the angle which it makes with the straight line aCb; then if we apply a force R in the direction CR, we may suppose the fulcrum removed, and the body to be held in equilibrium by the forces P, W, R.



We shall resolve these forces in directions parallel and perpendicular to aCb; and also take their moments about C; then by Art. 57 we have the following equations:

$$P\cos\alpha - W\cos\beta - R\cos\theta = 0 \quad \ldots\ldots\ldots\ldots(1),$$

$$P\sin\alpha + W\sin\beta - R\sin\theta = 0 \quad \ldots\ldots\ldots\ldots(2),$$

and

$$P.CD - W.CE = 0 \quad \ldots\ldots\ldots\ldots(3),$$

CD and CE being drawn perpendicular to the directions of P and W.

These three equations determine the ratio of P to W when there is equilibrium; and the magnitude and direction of the pressure on the fulcrum.

For equation (3) gives

$$\frac{P}{W} = \frac{CE}{CD} = \frac{\text{perpendicular on direction of } W}{\text{perpendicular on direction of } P} \quad \ldots\ldots(4).$$

Also by transposing the last terms of (1) and (2), we have

$$R\cos\theta = P\cos\alpha - W\cos\beta,$$

$$R\sin\theta = P\sin\alpha + W\sin\beta.$$

Add their squares; therefore

$$R^2 = P^2 + W^2 - 2PW\cos(\alpha+\beta),$$

which gives the magnitude of R.

From (1) and (2) by transposition and division

$$\tan\theta = \frac{P\sin\alpha + W\sin\beta}{P\cos\alpha - W\cos\beta},$$

which gives the direction of the pressure.

If we suppose B to be the fulcrum and take the moments about B instead of C, we have instead of equation (4) the following:

$$\frac{P}{R} = \frac{\text{perpendicular on direction of } R}{\text{perpendicular on direction of } P} \quad \ldots\ldots\ldots\ldots (5).$$

This is not a new equation of condition; but is a consequence of the three already given, (1), (2), (3). To shew this, imagine AD and BE produced to meet CR: they will meet this straight line at the same point, since the distances by these two constructions are CD cosec $(\theta-\alpha)$ and CE cosec $(\theta+\beta)$; and these are made equal, by equations (1), (2), (3), if we eliminate P and W. Suppose, then, F to be the point at which these straight lines meet. By multiplying (1), (2), respectively by $\sin\beta$ and $\cos\beta$, and adding, we have

$$\frac{P}{R} = \frac{\sin(\theta+\beta)}{\sin(\alpha+\beta)} = \frac{FB\sin(\theta+\beta)}{FB\sin(\alpha+\beta)}$$
$$= \frac{\text{perpendicular on direction of } R}{\text{perpendicular on direction of } P};$$

therefore this equation is a consequence of the equations (1), (2), (3), as might have been anticipated.

It follows, then, that *the condition of equilibrium in a lever of any species is that the two forces must be inversely as the perpendiculars drawn on their directions from the fulcrum and the forces must act so as to tend to turn the lever in opposite directions round the fulcrum.*

148. This property of the lever renders it a useful instrument in multiplying the efficacy of a force. For any

two forces, however unequal in magnitude, may be made to balance each other simply by fixing the fulcrum so that the ratio of its distances from the directions of the forces shall be equal to the inverse ratio of the forces. If the fulcrum be moved from this position, then that force will preponderate from which the fulcrum is moved and the equilibrium will be destroyed. We are thus led to understand how mechanical advantage is gained by using a crow-bar to move heavy bodies, as large blocks of stone: a poker to raise the coals in a grate: scissors, shears, nippers, and pincers; these last consisting of two levers of the first kind. The brake of a pump is a lever of the first kind. In the Stanhope printing-press we have a remarkable illustration of the mechanical advantage that can be gained by levers. The frame-work in which the paper to be printed is fixed, is acted on by the shorter arm of a lever, the other arm being connected with a second lever, the longer arm of which is worked by the pressman. These levers are so adjusted that at the instant the paper comes in contact with the types, the perpendiculars from the fulcra on the directions of the forces acting at the shorter arms are exceedingly short, and consequently the levers multiply the force exerted by the pressman to an enormous extent.

As examples of levers of the second kind, we may mention a wheelbarrow, a chipping-knife, a pair of nutcrackers.

It must be observed, however, that as the lever moves about the fulcrum the space through which the weight is moved is, in the second and usually in the first species of lever, smaller than the space passed through by the power: and therefore what is gained in power is lost in speed. For example in the case of the crow-bar: to raise a block of stone through a given space by applying the hand at the further extremity of the lever, we must move the hand through a greater space than that which the weight describes.

But in the third species of lever the reverse is the case. The power is nearer the fulcrum than the weight, and is consequently greater; but the motion of the weight is greater than that of the power. In this kind of lever speed is gained at the expense of power. An excellent example is the treddle of a turning lathe. But the most striking example of levers of the third kind is found in the animal frame,

in the construction of which it seems to be a prevailing principle to sacrifice power to readiness and quickness of action. The limbs of animals are generally levers of this description. The condyle of the bone rests in its socket as the fulcrum; a strong muscle attached to the bone near the condyle is the power, and the weight of the limb together with any resistance opposed to its motion is the weight. A slight contraction of the muscle gives a considerable motion to the limb.

149. The lever is applied to determine the weight of substances. Under this character it is called a Balance. The Common Balance has its two arms equal, with a scale suspended from each extremity; the fulcrum being slightly *above* the centre of gravity of the beam and also slightly above the line joining the two points from which the scales are suspended. The substance to be weighed is placed in one scale, and weights placed in the other till the beam remains in equilibrium in a perfectly horizontal position; in which case the weight of the substance is indicated by the weights by which it is balanced. If the weights differ ever so slightly the horizontality of the beam will be disturbed, and after oscillating for some time, in consequence of the fulcrum being in the position above described, it will, on attaining a state of rest, form an angle with the horizon, the extent of which is a measure of the sensibility of the balance.

When we take the weight in the other scale as a measure of the weight of the substance we are weighing, we assume that the arms of the lever are of equal length and that the beam would be itself in equilibrium if the scales were empty. We can ascertain if these conditions are satisfied by observing whether equilibrium still subsists when the substance is transferred to the scale which the weight originally occupied and the weight to that which the substance originally occupied.

150. In the construction of a balance the following requisites are to be considered.

(1) When loaded with equal weights the beam should be perfectly horizontal.

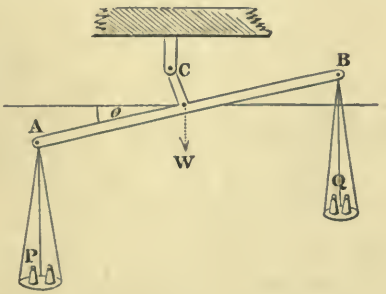
(2) When the weights differ, even by a slight quantity, the *sensibility* should be such as to detect this difference.

(3) When the balance is disturbed it should quickly return to its position of equilibrium, or it should have *stability*.

We shall now consider how these requisites may be satisfied.

To find how the requisites of a good balance may be satisfied.

Let P and Q be the weights in the scales; A and B the points of suspension of the scales; let $AB = 2a$. Let



C be the fulcrum, h its distance from the straight line which joins A and B. Let W be the weight of the beam, k the distance of the centre of gravity of the beam from C; this centre of gravity being supposed to lie on the perpendicular from C on the straight line which joins A and B. Let S be the weight of each scale; so that P and S act vertically through A, and Q and S vertically through B. Let θ be the angle which the beam makes with the horizon when there is equilibrium.

The sum of the moments of the weights round C will be zero when there is equilibrium, by Art. 57. Now the length of the perpendicular from C on the line of action of P and S is $a\cos\theta - h\sin\theta$; the length of the perpendicular from C on the line of action of Q and S is $a\cos\theta + h\sin\theta$; and the length of the perpendicular from C on the line of action of W is $k\sin\theta$. Therefore

$$(Q+S)(a\cos\theta+h\sin\theta)-(P+S)(a\cos\theta-h\sin\theta)+Wk\sin\theta=0;$$

therefore $$\tan\theta = \frac{(P-Q)\,a}{(P+Q+2S)\,h+Wk}.$$

This determines the position of equilibrium. The first requisite—the horizontality when P and Q are equal—is satisfied by making the arms equal.

The *sensibility* is inversely proportional to the smallest difference of weight which the balance can detect compared with the total weight of the body. For our present purpose it may be measured by

$$\frac{P+Q}{P-Q}\tan\theta,$$

that is, by

$$\frac{(P+Q)\,a}{(P+Q+2S)\,h+Wk},$$

the reciprocal of which is

$$\left(1+\frac{2S}{P+Q}\right)\frac{h}{a}+\frac{W}{P+Q}\frac{k}{a}.$$

In order that this may be as small as possible, h and k should be small compared with a, and S and W should be small compared with P and Q.

The *stability* is greater the greater the moment of the forces which tend to restore equilibrium, for a given value of θ, when P and Q are equal, and the less the masses which have to be set in vibration. The moment in question is

$$\{(P+Q+2S)\,h+Wk\}\sin\theta.$$

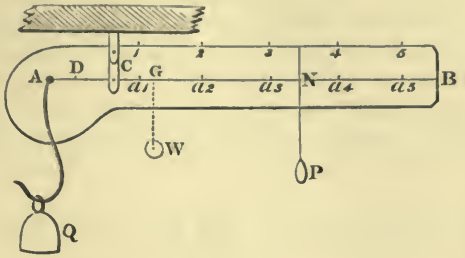
Hence for quick vibrations h and k should be large, a condition inconsistent with high sensibility.

151. Another kind of balance is that in which the arms are unequal, and the same weight is used to weigh different substances by varying its point of support, and observing its distance from the fulcrum by means of a graduated scale. The common steelyard is of this description.

152. *To shew how to graduate the common steelyard.*

Let AB be the beam of the steelyard. A the fixed point from which the substance to be weighed is suspended, Q being its weight; C the fulcrum; W the weight of the beam

together with the hook or scale-pan suspended from A; G the centre of gravity of these.



Suppose that P suspended at N balances Q suspended from A; then, taking the moments of P, Q, and W about C, we have

$$Q \,.\, AC - W \,.\, CG - P \,.\, CN = 0;$$

therefore

$$Q = \frac{CN + \frac{W}{P} \,.\, CG}{AC} P.$$

Take the point D, so that $CD = \frac{W}{P} CG$; therefore

$$Q = \frac{CN + CD}{AC} P = \frac{DN}{AC} P.$$

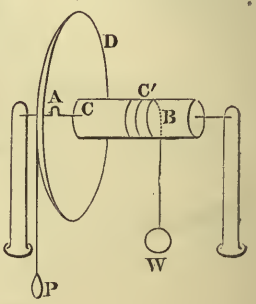
Now let the arm DB be graduated by taking Da_1, Da_2, Da_3,..... equal respectively to AC, $2AC$, $3AC$.....; let the figures 1, 2, 3, 4,..... be placed over the points of graduation, and let subdivisions be made between these. Then by observing the graduation at N we know the ratio of Q to P; and P being a given weight we know the weight of Q.

A better plan in practice would be to mark the two positions of the given weight P in which it balances two known weights Q_1 and Q_2 differing as widely as possible from each other. The values of Q_1 and Q_2 (say in lbs.) are to be engraved over these two marks, the distance between them is then to be divided into $Q_2 - Q_1$ equal parts, and the equidistant divisions are to be continued both ways.

153. The second of the Mechanical Powers is the Wheel and Axle. This machine consists of two cylinders fixed together, with their axes in the same straight line: the larger cylinder is called the wheel, and the smaller the axle. The cord by which the weight is suspended is fastened to the axle, and then coiled round it, while the power which supports the weight acts by a cord coiled round the circumference of the wheel, by spokes acted on by the hand, as in the *capstan*, or by the hand acting on a handle, as in the *windlass*.

154. *To find the ratio of the power and weight in the Wheel and Axle when in equilibrium.*

Let AD be the wheel and $CC'B$ the axle; P the power represented by a weight suspended from the circumference of the wheel at A; W the weight hanging from the axle at B.



Then since the axis of the machine is fixed, the condition of equilibrium is that the sum of the moments of the forces about this axis vanishes, (Art. 87); therefore

$P \times$ rad. of wheel $= W \times$ rad. of axle;

therefore $$\frac{W}{P} = \frac{\text{rad. of wheel}}{\text{rad. of axle}}.$$

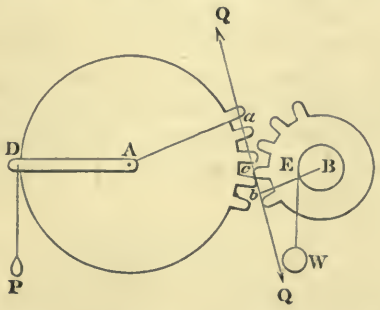
It will be seen that this machine is only a modification of the lever. In short it is an assemblage of levers all having the same axis: and as soon as one lever has been in action the next comes into play; and in this way an endless leverage is obtained. In this respect, then, the wheel and axle possesses a great advantage over the common lever. It is much used in docks and in shipping.

155. The third Mechanical Power is the Toothed Wheel. It is extensively applied in all machinery; in cranes, steam-engines, and particularly in clock and watch work. If two circular hoops of metal or wood having their outer circum-

ferences indented, or cut into equal teeth all the way round, be so placed that their edges touch, one tooth of one circumference lying between two of the other (as represented in the figure); then if one of them be turned round by any means, the other will be turned round also. This is the simple construction of a pair of toothed wheels.

156. *To find the relation of the power and weight in Toothed Wheels.*

Let A and B be the fixed centres of the toothed wheels



on the circumference of which the teeth are arranged; C the point of contact of two teeth; QCQ a normal to the surfaces in contact at C. [This line is incorrectly drawn in the figure; it ought to slope the opposite way.] Suppose an axle is fixed on the wheel B, and the weight W suspended from it at E by a cord; also suppose the power P acts by an arm AD; draw Aa, Bb perpendicular to QCQ. Let the mutual pressure at C be Q. Then, since the wheel A is in equilibrium about the fixed axis A, the sum of the moments about A equals zero; therefore

$$P \,.\, AD - Q \,.\, Aa = 0.$$

Also since the wheel B is in equilibrium about B, the sum of the moments about B equals zero; therefore

$$Q \,.\, Bb - W \,.\, BE = 0.$$

Then by eliminating Q from these two equations,

$$\frac{P}{W} = \frac{P}{Q} \cdot \frac{Q}{W} = \frac{Aa}{AD} \cdot \frac{BE}{Bb};$$

$$\text{or } \frac{\text{moment of } P}{\text{moment of } W} = \frac{Aa}{Bb};$$

when the teeth are small this ratio very nearly

$$= \frac{\text{rad. of wheel } A}{\text{rad. of wheel } B}.$$

Toothed wheels of the best class are so constructed that QQ cuts the line of centres AB in a *fixed* point X. Then since

$$\frac{Aa}{Bb} = \frac{AX}{BX}$$

the ratio of P to Q for equilibrium is constant, and the motion can be shewn to be the same as that of two circular wheels of radii AX and BX rolling together. If the arms AD and BE at which P and Q act are respectively equal to AX and BX, P and Q must be equal.

157. Wheels are in some cases turned by means of straps passing over their circumferences. In such cases the minute protuberances of the surfaces prevent the sliding of the straps, and a mutual action takes place such as to render the calculation exactly analogous to that in the Proposition.

For the calculation of the best forms for the teeth, the reader is referred to works which discuss the subject of machinery.

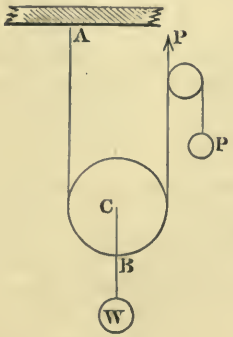
158. The fourth Mechanical Power is the Pully. There are several species of pullies: we shall mention them in order. The simple pully is a small wheel moveable about its axis: a string passes over part of its circumference. If the axis is fixed the effect of the pully is only to change the direction of the string passing over it: if however the axis be moveable, then, as will be presently seen, a mechanical advantage may be gained.

It is sometimes assumed as axiomatic that if a perfectly flexible string passes over a smooth surface the tension of the string will be the same throughout; we shall see, however, in

the Chapter on Flexible Strings that this result admits of demonstration. In the present Chapter we shall only require a part of the general proposition. We shall suppose the pullies to be circular, and assume that the tensions of the two portions of any string which are separated by a portion in contact with a pully are equal. And this may be shewn to be necessarily true if we merely admit that the string is a tangent to the circle at the point where it ceases to be in contact with the pully. For since the pully is smooth the directions of all the forces which it exerts on the string must pass through the *centre of the pully;* hence if we take the moments with respect to this point of the forces which act on the string we see that the string cannot be in equilibrium unless the tensions of the two portions are equal.

159. *To find the ratio of the power and weight in the single moveable Pully.*

I. Suppose the parts of the string divided by the pully are parallel.



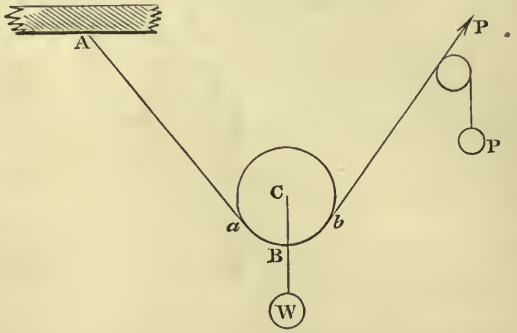
Let the string ABP have one extremity fixed at A, and after passing under the pully at B suppose it held by the hand exerting a force P, or it may be passed over a fixed pully. The weight W is suspended by a string from the centre C of the pully.

Now the tension of the string ABP is the same throughout. Hence the pully is acted on by three parallel forces, P, P,

and W; hence

$$2P - W = 0 : \text{ therefore } \frac{W}{P} = 2.$$

II. Suppose the portions of the string are not parallel.



Let α and α' be the angles which Aa and Pb make with the vertical.

Now the pully is held in equilibrium by W in CW, P in aA, and P in bP. Hence, resolving the forces horizontally and vertically,

$$P \sin \alpha - P \sin \alpha' = 0 \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$P \cos \alpha + P \cos \alpha' - W = 0 \ldots\ldots\ldots\ldots\ldots\ldots (2);$$

the equation of moments round C is an identical equation.

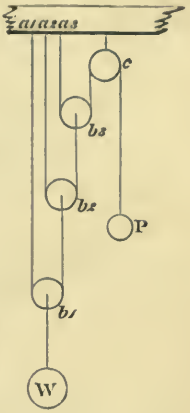
By (1), $\sin \alpha = \sin \alpha'$ and $\alpha = \alpha'$;

therefore, by (2), $$\frac{W}{P} = 2 \cos \alpha,$$

which is the relation required.

160. *To find the ratio of the power and weight in a system of pullies, in which each pully hangs from a fixed point by a separate string, one end being fastened in the pully above it and the other end on a fixed beam, and all the strings being parallel.*

Let n be the number of moveable pullies.



I. Let us neglect the weight of the pullies themselves. Then

$$\text{tension of } b_1 W = W;$$

$$\text{tension of } a_1 b_1 b_2 = \tfrac{1}{2} W;$$

$$\text{tension of } a_2 b_2 b_3 = \frac{1}{2^2} W;$$

$$\text{tension of } a_3 b_3 c = \frac{1}{2^3} W;$$

and so on; and the tension of the string passing under the n^{th} pully $= \dfrac{1}{2^n} W$, and this is equal to P; therefore

$$\frac{W}{P} = 2^n.$$

II. Let us suppose the weights of the pullies to be considered; and let $\omega_1, \omega_2, \omega_3, \ldots \omega_n$ be these weights.

Then if $p_1, p_2, p_3, \ldots p_n$ be the weights which they would sustain at P, and P_1 the weight which W would sustain at P, we have

$$p_1 = \frac{\omega_1}{2^n}, \; p_2 = \frac{\omega_2}{2^{n-1}}, \ldots\ldots p_n = \frac{\omega_n}{2}, \quad P_1 = \frac{W}{2^n};$$

therefore $$P = p_1 + p_2 + \ldots\ldots + p_n + P_1,$$

or $$P = \frac{1}{2^n}\left\{ W + \omega_1 + 2\omega_2 + 2^2\omega_3 + \ldots\ldots + 2^{n-1}\omega_n \right\}.$$

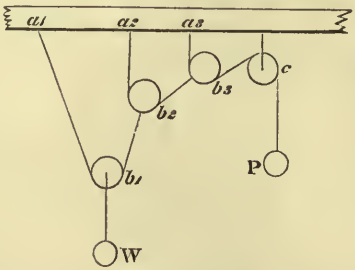
If $\omega_1 = \omega_2 = \omega_3 = \ldots\ldots = \omega_n$,

$$P = \frac{1}{2^n}\{W + (2^n - 1)\,\omega_1\}, \text{ that is } P - \omega_1 = \frac{1}{2^n}(W - \omega_1).$$

161. *To find the ratio of the power and weight when the system is the same as in the last Proposition, but the strings are not parallel.*

We shall neglect the weights of the blocks. The pullies will evidently so adjust themselves that the string at the centre of any pully will bisect the angle between the strings touching its circumference.

Let $2\alpha_1$, $2\alpha_2$, $2\alpha_3$, ... $2\alpha_n$ be the angles included between



the strings touching the first, second, third, ... n^{th} pullies respectively.

Then, by Art. 159,

$$\text{tension of } a_1b_1b_2 = \frac{W}{2\cos\alpha_1};$$

$$\text{tension of } a_2b_2b_3 = \frac{W}{2^2\cos\alpha_1\cos\alpha_2};$$

$$\text{tension of } a_3b_3c = \frac{W}{2^3\cos\alpha_1\cos\alpha_2\cos\alpha_3};$$

$$\text{tension of the last string} = \frac{W}{2^n\cos\alpha_1\cos\alpha_2\cos\alpha_3\ldots\cos\alpha_n},$$

and this is equal to P; therefore

$$\frac{W}{P} = 2^n\cos\alpha_1\cos\alpha_2\cos\alpha_3\ldots\cos\alpha_n.$$

162. *To find the relation of the power and weight in a system of pullies where the same string passes round all the pullies.*

This system consists of two blocks, each containing a number of the pullies with their axes coincident. The weight

is suspended from the lower block, which is moveable, and the power acts at the loose extremity of the string, which passes round the respective pullies of the upper and lower block alternately.

Since the same string passes round all the pullies, its tension will be everywhere the same, and equal to the power P. Let n be the number of portions of string at the lower block; then $n \,.\, P$ will be the sum of their tensions; therefore

$$W = n \,.\, P.$$

If we take into account the weight of the lower block, and call it B, then

$$W + B = n \,.\, P.$$

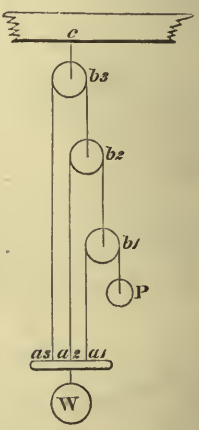
If the strings at the lower block are not vertical, we must take the sum of the parts resolved vertically, and equate it to W. But in general this deviation from the vertical is so slight that it is neglected.

163. As the weight is rising or falling, it will be observed that in general the pullies move with different angular motions. The degree of angular motion of each pully depends upon the magnitude of its radius. Mr James White took advantage of this, to choose the radii of the pullies in such a manner as to give those in the same block the same angular motion, and so to prevent the wear and resistance caused by the friction of the pullies against each other. This being the case, the pullies in each block might be fastened together, or, instead of this, cut out of one mass.

It will be seen without much difficulty, that if the weight W be raised through a space a, each of the portions of string between the two blocks will be shortened by the length a; and therefore, that the portions of string which move over the pullies in the two blocks, taken alternately, will have their lengths equal to a, $2a$, $3a$, $4a$... Suppose the end of the string fastened to the lower block; then if the radii of the pullies of the upper block be proportional to the odd numbers 1, 3, 5, ... these pullies will move with the same

angular velocity, and might be made all in one piece, as mentioned above. And if the radii of the lower pullies be proportional to the even integers 2, 4, 6, ... these also will move with a common angular velocity, and might therefore be cut out of one piece.

164. *To find the ratio of the power to the weight when all the strings are attached to the weight.*



If we neglect the weights of the pullies, the tension of the string $b_1a_1 = P$; the tension of $a_2b_2 = 2P$; and so on: if there be n pullies, then the sum of the tensions of the strings attached to the weight

$$= P + 2P + 2^2P + \ldots + 2^{n-1}P = (2^n - 1)P;$$

$$\text{therefore } \frac{W}{P} = 2^n - 1.$$

If we suppose the weights of the pullies are $\omega_1, \omega_2, \omega_3, \ldots$ reckoning from the lowest, and $\omega', \omega'', \omega''', \ldots$ the portions of W which they respectively support, since they evidently assist P, and W' the portion of W supported by P; then

$$W' = (2^n - 1)P,$$

$$\omega' = (2^{n-1} - 1)\omega_1,$$

$$\omega'' = (2^{n-2} - 1)\omega_2,$$

$$\ldots\ldots\ldots\ldots\ldots$$

$$\omega^{(n-1)} = (2 - 1)\omega_{n-1};$$

therefore $W = W' + \omega' + \ldots\ldots = (2^n - 1)P + (2^{n-1} - 1)\omega_1$

$$+ (2^{n-2} - 1)\omega_2 + \ldots\ldots + (2 - 1)\omega_{n-1}.$$

If $\omega_1 = \omega_2 = \omega_3 = \ldots\ldots$

$$W = (2^n - 1)P + \{2^{n-1} + 2^{n-2} + \ldots\ldots + 2 - (n-1)\}\omega_1,$$

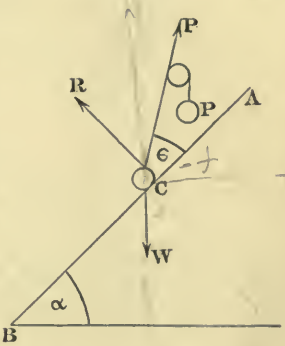
$$= (2^n - 1)P + (2^n - n - 1)\omega_1.$$

165. The fifth Mechanical Power is the Inclined Plane.

By an inclined plane we mean a plane inclined to the horizon. A weight W may be supported on an inclined plane by a power P less than W.

166. *To find the ratio of the power and weight in the inclined plane.*

Let AB be the inclined plane; α the angle which it makes with the horizon. Let the power P act on the weight in the direction CP, making an angle ϵ with the plane. Now the weight at C is held at rest by P in CP, W in the vertical CW, and a pressure R in CR, at right angles to the plane.



Hence, by Art. 27, if we resolve these forces perpendicular and parallel to the plane, we have

$$R + P\sin\epsilon - W\cos\alpha = 0 \quad \ldots\ldots\ldots\ldots\ldots(1),$$

$$P\cos\epsilon - W\sin\alpha = 0 \quad \ldots\ldots\ldots\ldots\ldots(2).$$

The second equation gives the required relation $\frac{P}{W} = \frac{\sin\alpha}{\cos\epsilon}$; and the first equation gives the magnitude of the pressure R.

If P act horizontally, $\epsilon = -\alpha$, and $P = W\tan\alpha$.

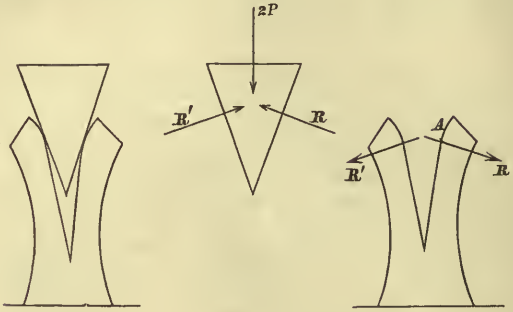
If P act parallel to the plane, $\epsilon = 0$, and $P = W\sin\alpha$.

If P act vertically, $\epsilon = \frac{1}{2}\pi - \alpha$, and $P = W$.

167. The sixth Mechanical Power is the Wedge. This is a triangular prism, and is used to separate obstacles by introducing its edge between them and then thrusting the wedge forward. This is effected by the blow of a hammer or other such means, which produces a violent pressure for a short time, sufficient to overcome the greatest forces.

168. *An isosceles wedge is kept in equilibrium by pressures on its three faces; to find the relation between them.*

The three figures below represent the wedge and obstacles together and separately.



Let $2P$ denote the force acting perpendicularly to the thick end of the wedge; R and R' the forces which act on the other faces of the wedge: these forces are perpendicular to the faces since the wedge is supposed smooth.

Let 2α be the vertical angle of the wedge.

Resolve the forces which act on the wedge in directions perpendicular and parallel to the thick end; then for the equilibrium of the wedge we have

$$2P = (R + R') \sin \alpha,$$

$$R \cos \alpha = R' \cos \alpha;$$

therefore

$$R = R',$$

$$P = R \sin \alpha.$$

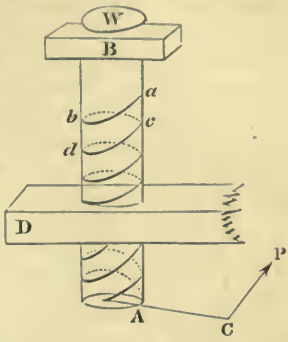
The point A of the obstacle in contact with the wedge may be regarded as a particle constrained to move in a given curve. Let S denote the resistance to its displacement along the curve, and i the angle at which R cuts the curve. Then we must have

$$R \cos i = S,$$

hence

$$\frac{P}{S} = \frac{\sin \alpha}{\cos i}.$$

169. The last "Mechanical Power" is the Screw. It consists of a cylinder AB with a uniform projecting thread $abcd\ldots$ traced round its surface, and making a constant angle with straight lines parallel to the axis of the cylinder. This cylinder fits into a block D pierced with an equal cylindrical aperture, on the inner surface of which is cut a groove the exact counterpart of the projecting thread $abcd\ldots$



It is easily seen from this description, that when the cylinder is introduced into the block, the only manner in which it can move is backwards or forwards by revolving about its axis, the thread sliding in the groove. Suppose W to be the weight acting on the cylinder, including the weight of the cylinder itself, and P to be the power acting at the end of an arm AC at right angles to the axis of the cylinder; the block D is supposed to be firmly fixed, and the axis of the cylinder to be vertical.

170. *To find the ratio of the power and weight in the Screw when they are in equilibrium.*

Let the distance of C from the axis of the cylinder $= a$; and the radius of the cylinder $= b$.

The forces which hold the screw in equilibrium are W, P, and the reactions against the surface of the thread.

The thread is bounded externally and internally by two cylinders with a common axis, and we shall first suppose its form to be such as would be generated by the motion of a rectangle having one pair of sides vertical and the other pair horizontal, the plane of the rectangle being supposed always to pass through the axis. We shall assume the external and internal radii to be so nearly equal that b may denote either of them indifferently. Let $\frac{\pi}{2} - \alpha$ be the angle which the thread makes with the axis, then α is the angle

which the direction of the reaction at a point of either the upper or lower face makes with the axis. If, then, R be one of these reactions, reckoned positive when it is exerted against the under face and negative when against the upper, $R\cos\alpha$, $R\sin\alpha$ are its vertical and horizontal components, and the horizontal components act at right angles to the radii at their respective points of application. Hence, resolving the forces vertically, and also taking moments round the axis, we have

$$W = \Sigma R\cos\alpha = \cos\alpha\,\Sigma R \quad \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$Pa = \Sigma Rb\sin\alpha = b\sin\alpha\,\Sigma R \quad \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

The reactions against the cylindrical surfaces do not enter into equation (1) because they are horizontal, and do not enter into equation (2) because they give no moments round the axis.

Hence $$\frac{W}{P} = \frac{a\cos\alpha}{b\sin\alpha} = \frac{2\pi a}{2\pi b\tan\alpha}$$

$$= \frac{\text{circumference of circle of which the radius is } a}{\text{vertical dist. between two successive winds of the thread}}.$$

If the thread is not rectangular, the reaction at any point of it can be resolved into two components having the same directions as the reactions above discussed. The component which intersects the axis at right angles can be neglected, and if the other component be denoted by R, equations (1) and (2) will still hold.

The Screw is used to gain mechanical power in many ways. In excavating the Thames Tunnel, the heavy iron frame-work which supported the workmen was gradually advanced by means of large screws.

MISCELLANEOUS EXAMPLES.

1. If one arm of a common balance be longer than the other, shew that the real weight of any body is the geometrical mean between its apparent weights as weighed first in one scale and then in the other.

2. The arms of a false balance are unequal, and one of the scales is loaded; a body whose true weight is P lbs. appears to weigh W lbs. when placed in one scale, and W' lbs. when placed in the other scale: find the ratio of the arms and the weight with which the scale is loaded.

Results. $\dfrac{W'-P}{P-W}$; $\dfrac{WW'-P^2}{P-W}$.

3. A triangular lamina ABC, whose weight is W, is suspended by a string fastened at C: find the weight which must be attached at B that the vertical through C may bisect the angle ACB.

Result. $\dfrac{W}{3}\dfrac{b-a}{a}$.

4. Two equal weights are suspended by a string passing freely over three tacks, which form an isosceles triangle whose base is horizontal: find the vertical angle when the pressure on each tack is equal to one of the weights. *Result.* 120°.

5. A uniform heavy rod, at a given point of which a given weight is attached, is sustained at one end; determine its length when the force which applied at the other end will keep it horizontal is least.

6. $ABGC$, DEF are two horizontal levers without weight; B, F their fulcrums: the end D of one lever rests on the end C of the other; HK is a rod without weight suspended by two equal parallel strings from the points E, G. Prove that a weight P at A will balance a weight W placed anywhere on the rod HK, provided

$$\frac{EF}{DF}=\frac{BG}{BC} \text{ and } \frac{P}{W}=\frac{BG}{AB}.$$

7. If the axis about which a wheel and axle turns coincide with that of the axle, but not with the axis of the wheel, find the greatest and least ratios of the power and weight necessary for equilibrium, neglecting the weight of the machine.

8. In the system of pullies where each string is attached to the weight, let one of the strings be nailed to the block through which it passes, then shew that the power may be increased up to a certain limit without producing motion. If there be three pullies, and the action of the middle one be checked in the manner described, find the tension of each string for given values of P and W.

9. A weight w is supported on an inclined plane by two forces, each equal to $\frac{w}{n}$, one of which acts parallel to the base. Shew that equilibrium may be possible when the inclination of the plane is not greater than $2\tan^{-1}\left(\frac{1}{n}\right)$, n being a positive integer.

10. A weight is suspended from the two ends of a straight lever without weight whose length is 5 feet, by strings whose lengths are 3 and 4 feet. Find the position of the fulcrum that the lever may rest in a horizontal position.

Result. At a distance $3\frac{1}{5}$ feet from that end of the lever to which the longer string is fastened.

11. A uniform steelyard AB, having a constant weight P, and a scale-pan of weight kP, suspended at B and A respectively, is used as a balance by moving the rod backwards and forwards upon the fulcrum C, on which the whole rests. Shew that the beam must be graduated by the formula

$$AC = \frac{1 + \frac{1}{2}k'}{n + k + k' + 1} \cdot AB,$$

the weight of the rod being $k'P$, and n being each of the natural numbers 1, 2, 3, ... taken in succession.

12. AB is a rod without weight capable of turning freely about its extremity A, which is fixed; CD is another rod equal to $2AB$, and attached at its middle point to the extremity B of the former, so as to turn freely about this point: a given force acts at C in the direction CA, find the force which must be applied at D in order to produce equilibrium.

13. A lever without weight in the form of the arc of a circle, having two weights P and Q suspended from its extremities, rests with its convexity downwards on a horizontal plane: determine the position of equilibrium.

Result. Let α be the angle which the arc subtends at the centre of the circle, θ the inclination to the vertical of the radius at the extremity of which P is suspended; then

$$\tan\theta = \frac{Q \sin\alpha}{P + Q\cos\alpha}.$$

14. The sides of a rhombus $ABCD$ are hinged together at the angles; at A and C are two pulling forces (P, P) acting in the diagonal AC; and at B and D there are two other pulling forces (Q, Q) acting in BD: shew that

$$\cos BAD = \frac{P^2 - Q^2}{P^2 + Q^2}.$$

15. AB, BC are two equal and uniform beams connected by a hinge at B; there is a fixed hinge at A; a string fastened at C passes over a pully at D and is attached to a weight P; AD is horizontal and equal to twice the length of either beam: shew that if P be such as to keep BC horizontal $P = W.\sqrt{\frac{3}{2}}$, and $\tan\theta = 2\tan\phi = 2\sqrt{2}$, where θ and ϕ are the angles which AB, CD make with the horizon, and $2W$ the weight of each beam.

16. A string $ABCDEP$ is attached to the centre A of a pully whose radius is r; it then passes over a fixed point B and under the pully which it touches at the points C and D; it afterwards passes over a fixed point E and has a weight P attached to its extremity; BE is horizontal and $= \frac{8}{3}r$, and DE is vertical: shew that if the system be in equilibrium the weight of the pully is $\frac{5}{2}P$, and find the distance AB.

Result. $$AB = \frac{8r}{3\sqrt{7}}.$$

17. Three uniform rods rigidly connected in the form of a triangle rest on a smooth sphere of radius r; shew that the tangent of the inclination of the plane of the triangle to the

horizon is $\frac{a}{\sqrt{(r^2-\rho^2)}}$, where a is the distance of the centres of the circles inscribed in the triangle itself and in the triangle formed by joining the middle points of the rods, and ρ is the radius of the circle inscribed in the triangle.

18. If a steelyard be constructed with a given rod whose weight is inconsiderable compared with that of the sliding weight, the sensibility varies inversely as the sum of the sliding weight and the greatest weight which can be weighed.

19. A heavy equilateral triangle hung up on a smooth peg by a string, the ends of which are attached to two of its angular points, rests with one of its sides vertical: shew that the length of the string is double the height of the triangle.

20. Three equal heavy spheres lying in contact on a horizontal plane are held together by a string which passes round them. A cube, whose weight is W, is placed with one of its diagonals vertical so that its lower faces touch the spheres; shew that the tension of the string is not less than $\frac{W\sqrt{2}}{3\sqrt{3}}$.

21. A roof of given span is to be constructed of two beams, which are to be connected at the vertex by a single pin, and the weight of the roof would increase in proportion to the length of the beams: find the angle of inclination to the horizon, when the whole pressure on the wall is the least possible.

Shew that the direction of the line of pressure will then make the same angle with the vertical line which the beam makes with the horizontal line.

22. An endless string supports a system of equal heavy pullies, the highest of which is fixed, the string passing round every pully and crossing itself between each. If α, β, γ, &c. be the inclinations to the vertical of the successive portions of string, prove that $\cos\alpha$, $\cos\beta$, $\cos\gamma$, &c. are in arithmetical progression.

23. Three equal heavy cylinders, each of which touches the other two, are bound together by a string and laid on a

horizontal table so that their axes are horizontal: the tension of the string being given, find the pressures between the cylinders.

Results. The horizontal pressure $= T - \frac{W}{2\sqrt{3}}$, the other $= T + \frac{W}{\sqrt{3}}$; where T is the tension of the string, and W the weight of each cylinder.

24. A string of equal spherical beads is placed upon a smooth cone having its axis vertical, the beads being just in contact with each other, so that there is no mutual pressure between them. Find the tension of the string; and deduce the limiting value when the number of beads is indefinitely great.

25. A smooth cylinder is supported on an inclined plane with its axis horizontal, by means of a string which, passing over the upper surface of the cylinder, has one end attached to a fixed point and the other to a weight W which hangs freely: if α be the inclination of the plane to the horizon, and θ the inclination to the vertical of that part of the string which is fastened to the fixed point, the weight of the cylinder is

$$2W \frac{\sin \frac{1}{2}\theta \cos (\alpha + \frac{1}{2}\theta)}{\sin \alpha}.$$

26. An inextensible string binds tightly together two smooth cylinders whose radii are given: find the ratio of the pressure between the cylinders to the tension by which it is produced.

Result. $\frac{4\sqrt{(r_1 r_2)}}{r_1 + r_2}$; where r_1 and r_2 are the given radii.

27. A ring whose weight is P is moveable along a smooth rod inclined to the horizon at an angle α; another ring of weight P' is moveable along a rod in the same vertical plane as the former and inclined at an angle α' to the horizon; a string which connects these rings passes through a third ring of weight $2W$; shew that the system cannot be in equilibrium unless

$$P \tan \alpha - P' \tan \alpha' + W (\tan \alpha - \tan \alpha') = 0.$$

28. A ball of given weight and radius is hung by a string of given length from a fixed point, to which is also attached another given weight by a string so long that the weight hangs below the ball: find the angle which the string to which the ball is attached makes with the vertical.

Result. Let Q be the weight of the ball, P the weight which hangs below the ball, a the radius of the ball, l the length of the string; then the inclination of the string to the vertical is $\sin^{-1}\left(\frac{P}{P+Q}\cdot\frac{a}{a+l}\right)$.

29. A right cone whose axis is a and vertical angle is $2\sin^{-1}\sqrt{\left(\frac{3}{7}\right)}$ is placed with its base in contact with a smooth vertical wall, and its curved surface on a smooth horizontal rod parallel to the wall: shew that it will remain at rest if the distance of the rod from the wall be not greater than a nor less than $\frac{a}{7}$.

30. A paraboloid is placed with its vertex downwards and axis vertical between two planes each inclined to the horizon at an angle of 45^0: find the greatest ratio which the height of the paraboloid may have to its latus rectum, so that if it be divided by a plane through its axis and the line of intersection of the inclined planes, the two parts may remain in equilibrium; also find the least ratio.

Result. Let h be the height and $4a$ the latus rectum; then the greatest and least ratios are determined respectively by

$$3a = \frac{32}{15\pi}\sqrt{(ah)}, \qquad h + \frac{32}{15\pi}\sqrt{(ah)} = 3a.$$

31. Three bars of given length are maintained in a horizontal position, and tied together at their extremities so as to form a horizontal triangle; and a smooth sphere of given weight and size rests upon them. Find the pressure of the sphere on each bar.

32. One end of a string is fastened to a point in a smooth vertical wall, the other to a point in the circumference of the base of a cylinder; the cylinder is in equilibrium, having a point of its upper end in contact with the wall: find the distance of this point below the point in the wall to which the string is fastened.

Result. Suppose x the required distance, l the length of the string, h the height of the cylinder, b the diameter of its base; then

$$3x^2 = l^2 - h^2 - b^2.$$

33. The ends of a string are fastened to two fixed points, and from knots at given points in the string given weights are hung; shew that the horizontal component of the tension is the same for all the portions into which the string is divided by the knots. Shew also that if the weights are all equal the tangents of the angles which the successive portions of the string make with the horizon are in Arithmetical Progression. (Such a system is called a *Funicular Polygon.*)

34. Two uniform beams loosely jointed at one extremity are placed upon the smooth arc of a parabola, whose axis is vertical and vertex upwards. If l be the semi-latus rectum of the parabola, and a, b the lengths of the beams, shew that they will rest in equilibrium at right angles to each other, if

$$l(a+b)(a^4+b^4)^{\frac{3}{2}} = a^4b^4;$$

and find the position of equilibrium.

35. A quadrilateral is formed by four rigid rods jointed at the ends; shew that two of its sides must be parallel in order that it may preserve its form when the middle points of either pair of opposite sides are joined together by a string in a state of tension.

36. Four rods, jointed at their extremities, form a quadrilateral, which may be inscribed in a circle: if they be kept in equilibrium by two strings joining the opposite angular points, shew that the tension of each string is inversely proportional to its length.

37. Four equal and uniform heavy rods being jointed by hinges so as to form a square, two opposite angles are connected by a string; this frame-work stands on a fixed point, the string being horizontal: find the tension of the string.

Result. Twice the weight of a rod.

38. Four equal and uniform heavy rods are connected by hinges; the system is suspended by a string attached to one hinge, and the lowest hinge is in contact with a horizontal plane: find the tension of the string and the pressure on the plane.

Result. Each is twice the weight of a rod.

39. A regular hexagon, composed of six equal heavy rods moveable about their angular points, is suspended from one angle which is connected by threads with each of the opposite angles. Shew that the tensions of the threads are as $\sqrt{3} : 2$. Find also the stress along each rod.

40. A regular hexagon is composed of six equal heavy rods moveable about their angular points; one rod is fixed in a horizontal position, and the ends of this rod are connected by vertical strings with the ends of the lowest rod: find the tension of each string.

Result. $\frac{3}{2} W$; where W is the weight of a rod.

41. Suppose that in the preceding Example each end of the fixed rod is connected with the more remote end of the lowest rod, so that the strings instead of being parallel are inclined at an angle of $60°$: find the tension of each string.

Result. $W\sqrt{3}$.

42. A regular hexagon is composed of six equal heavy rods moveable about their angular points, and two opposite angles are connected by a horizontal string; one rod is placed on a horizontal plane, and a weight is placed at the middle point of the highest rod: find the tension of the string.

Result. Let W be the weight of each rod, and W' the weight placed on the highest rod; then the tension is

$$\frac{3W + W'}{\sqrt{3}}.$$

CHAPTER X.

FRICTION.

171. In the investigations of the preceding Chapter, we have supposed that the surfaces of the bodies in contact are perfectly smooth. By a *smooth* surface is meant a surface which opposes no resistance whatever to the motion of a body upon it. A surface which does oppose a resistance to the motion of a body upon it is said to be *rough*. In practice it is found that all bodies are more or less rough.

The limiting friction of a body on a surface is measured by the least force which will put the body in motion along the surface. Friction acts in the direction opposite to that in which motion is about to take place.

172. Coulomb made a series of experiments upon the friction of bodies against each other and deduced the following laws. *Mémoires.....des Savans étrangers*, Tom. x.

(1) *The friction varies as the normal pressure when the materials of the surfaces in contact remain the same.* When the pressures are very great indeed, it is found that the friction is somewhat less than this law would give.

(2) *The friction is independent of the extent of the surfaces in contact so long as the normal pressure remains the same.* When the surfaces in contact are extremely small, as for instance a cylinder resting on a surface, this law gives the friction much too great.

These two laws are true when the body is on the point of moving and also when it is actually in motion; but in the case of motion the magnitude of the friction is not always the same as when the body is in a state bordering on motion: when there is a difference the friction is greater in the state bordering on motion than in actual motion.

(3) *The friction is independent of the velocity when the body is in motion.*

It follows from these laws that if P be the normal pressure between two surfaces, then the friction is μP, where μ is a constant quantity for the same materials and is called the *coefficient of friction.*

The following results, selected from a table given by Professor Rankine, will afford an idea of the amount of friction as determined by experiment; these results apply to the friction of motion.

For iron on stone	μ varies between ·3 and ·7.
For timber on timber	·2 and ·5.
For timber on metals	·2 and ·6.
For metals on metals	·15 and ·25.

For full particulars on this subject we refer the reader to Coulomb's papers, and to the Memoirs published in the *Mémoires de l'Institut*, by M. Morin; see also Rankine's *Manual of Applied Mechanics*, and Moseley's *Mechanical Principles of Engineering and Architecture.*

173. *Angle of Friction.* Suppose a body acted on only by its weight to be placed on a horizontal plane and the plane to be turned round a horizontal line until the body begins to slide. Let W be the weight of the body and α the angle the plane makes with the horizon. The pressure of the body on the plane will be equal to the resolved part of its weight perpendicular to the plane, that is to $W \cos \alpha$. The friction is equal to the resolved part of the weight parallel to the plane, that is to $W \sin \alpha$. If μ be the coefficient of friction, we have

$$W \sin \alpha = \mu W \cos \alpha;$$

therefore

$$\tan \alpha = \mu.$$

This experiment will enable us to determine the value of the coefficient of friction for different substances. The inclination of the plane when the body is just about to slide is called the *angle of friction.* That is to say; *the angle of friction between two surfaces is the greatest angle that their mutual pressure can make with their common normal without sliding taking place.*

174. In Art. 32 we have found the condition of equilibrium of a particle constrained to rest on a *smooth* curve; we proceed to the case of a particle on a *rough* curve. Suppose the curve a plane curve; let X, Y be the forces which act on the particle parallel to the axes of x and y exclusive of the action of the curve. The sum of the resolved parts of X and Y along the tangent to the curve is

$$X\frac{dx}{ds}+Y\frac{dy}{ds}.$$

The sum of the resolved parts along the normal is

$$\pm\left(X\frac{dy}{ds}-Y\frac{dx}{ds}\right).$$

If μ be the coefficient of friction the greatest friction capable of being called into action is

$$\pm\mu\left(X\frac{dy}{ds}-Y\frac{dx}{ds}\right).$$

Hence, the condition of equilibrium will be that the *numerical* value of $X\frac{dx}{ds}+Y\frac{dy}{ds}$ must be less than the *numerical* value of $\mu\left(X\frac{dy}{ds}-Y\frac{dx}{ds}\right)$, without regard to sign in either case. This may be conveniently expressed thus,

$\left(X\frac{dx}{ds}+Y\frac{dy}{ds}\right)^2$ must be less than $\mu^2\left(X\frac{dy}{ds}-Y\frac{dx}{ds}\right)^2$.

We may exhibit this condition in a different form, as will be seen in the following Article.

175. Next let the curve be not plane. Let P denote the resultant force acting on the particle exclusive of the action of the curve; X, Y, Z the components of P parallel to the axes; l, m, n the direction cosines of the tangent to the curve at the point where the particle is placed; θ the angle between this tangent and the direction of P. The resolved part of P along the tangent is $P\cos\theta$, and that at right angles to the tangent is $P\sin\theta$. Hence, if μ be the coefficient of friction, we must have for equilibrium

$$P\cos\theta<\mu P\sin\theta;$$

therefore $$\cos^2\theta < \mu^2(1-\cos^2\theta);$$

therefore $$\cos^2\theta < \frac{\mu^2}{1+\mu^2};$$

therefore $$\left(\frac{Xl+Ym+Zn}{P}\right)^2 < \frac{\mu^2}{1+\mu^2}.$$

It is easy to shew that this result includes that of the former Article by putting $n=0$, $m=\frac{dy}{ds}$, $l=\frac{dx}{ds}$.

176. *A particle is constrained to remain on a rough surface: determine the condition of equilibrium.*

Let P be the resultant force on the particle exclusive of the action of the surface; ϕ the angle between the direction of P and the normal to the surface at the point where the particle is placed; $u=0$ the equation to the surface; x, y, z the co-ordinates of the particle. The resolved part of P along the normal is $P\cos\phi$, and that at right angles to the normal is $P\sin\phi$. Hence, for equilibrium we must have

$$P\sin\phi < \mu P\cos\phi;$$

therefore $$\sin^2\phi < \mu^2\cos^2\phi;$$

therefore $$\cos^2\phi > \frac{1}{1+\mu^2};$$

therefore $$\frac{\left(X\frac{du}{dx}+Y\frac{du}{dy}+Z\frac{du}{dz}\right)^2}{P^2\left\{\left(\frac{du}{dx}\right)^2+\left(\frac{du}{dy}\right)^2+\left(\frac{du}{dz}\right)^2\right\}} > \frac{1}{1+\mu^2}.$$

177. In the following Articles of this Chapter we shall investigate certain equations which hold when the equilibrium of different machines is on the point of being disturbed. The equations in such cases will involve the forces acting on the machine and μ the coefficient of friction. When we have found one of these *limiting equations*, we can draw the following inferences:

(1) If in order to satisfy the equation for a given set of forces it is necessary to ascribe to μ a value greater than its

extreme value for the substances in question, which is known by experiment, equilibrium is impossible.

(2) If the limiting equation can be satisfied by ascribing to μ values less than its extreme value, equilibrium *may* be possible. We say *may* be possible, because the limiting equation may not be the *only* equation of equilibrium, and of course for equilibrium it is necessary that *all* the appropriate equations be satisfied.

We may illustrate the subject of friction by the solution of the following problem:

A weight W is placed on a rough horizontal plane; a string is attached to W and passes over a fixed smooth pully, and to the other end of the string a weight P is attached. Determine the limiting inclinations to the vertical of the string which passes from W to the fixed pully, when there is equilibrium.

Let θ be the inclination of this string to the vertical. The tension of the string is equal to P. The body on the rough horizontal plane is acted on by the tension of the string, by its own weight, by the normal resistance of the plane, and by the friction along the plane: denote the normal resistance by R, and the friction by μR.

Then resolving the forces horizontally and vertically we have

$$\mu R = P \sin \theta,$$

$$R + P \cos \theta = W.$$

Hence, eliminating R, we have

$$\mu (W - P \cos \theta) = P \sin \theta;$$

therefore

$$\mu = \frac{P \sin \theta}{W - P \cos \theta}$$

$$= \frac{\sin \theta}{k - \cos \theta},$$

where k is put for $\frac{W}{P}$.

We may, without any real loss of generality, suppose that θ is a positive angle not exceeding a right angle.

It may be shewn that the differential coefficient of $\frac{\sin\theta}{k-\cos\theta}$ with respect to θ is $\frac{k\cos\theta-1}{(k-\cos\theta)^2}$; this result will be useful as we proceed.

I. Suppose W less than P, so that k is less than unity.

Now μ may have any value from zero up to a certain limit, known by experiment, which we shall denote by tan ϵ. Thus at the limit

$$\tan\epsilon=\frac{\sin\theta}{k-\cos\theta},$$

therefore
$$\cot\epsilon+\cot\theta=\frac{k}{\sin\theta},$$

therefore
$$\sin(\theta+\epsilon)=k\sin\epsilon.$$

Let α be the least angle which has $k\sin\epsilon$ for its sine, so that

$$\sin(\theta+\epsilon)=\sin\alpha.$$

And as α is less than ϵ the only solution admissible here is

$$\theta+\epsilon=\pi-\alpha.$$

The expression $\frac{\sin\theta}{k-\cos\theta}$ is not positive unless θ is greater than the value which makes $\cos\theta=k$; and for greater values of θ the expression decreases as θ increases, and has its least value with which we are concerned when $\theta=\frac{\pi}{2}$; its value then being $\frac{1}{k}$, that is, $\frac{P}{W}$.

Hence we have the following results:

If the coefficient of friction is less than $\frac{P}{W}$ there is no position of equilibrium.

If the coefficient of friction is greater than $\frac{P}{W}$ equilibrium will subsist for all values of θ between $\pi - \alpha - \epsilon$ and $\frac{\pi}{2}$.

II. Suppose W greater than P, so that k is greater than unity.

In this case α is greater than ϵ, and the equation

$$\sin(\theta + \epsilon) = \sin\alpha$$

has two solutions which may be admissible, namely,

$$\theta + \epsilon = \alpha, \text{ and } \theta + \epsilon = \pi - \alpha.$$

The expression $\frac{\sin\theta}{k - \cos\theta}$ is always positive, and as θ increases from 0 to $\frac{\pi}{2}$ the expression increases up to a maximum value and then decreases. The maximum value is when $\cos\theta = \frac{1}{k}$, and is $\frac{1}{\sqrt{(k^2-1)}}$, that is, $\frac{P}{\sqrt{(W^2-P^2)}}$.

Hence we shall obtain the following results:

If the coefficient of friction is not less than $\frac{P}{\sqrt{(W^2-P^2)}}$, equilibrium will subsist for all values of θ between 0 and $\frac{\pi}{2}$.

If the coefficient of friction is less than $\frac{P}{W}$, equilibrium will subsist for all values of θ between 0 and $\alpha - \epsilon$.

If the coefficient of friction lies between $\frac{P}{W}$ and $\frac{P}{\sqrt{(W^2-P^2)}}$, equilibrium will subsist for all values of θ between 0 and $\alpha - \epsilon$, and between $\pi - \alpha - \epsilon$ and $\frac{\pi}{2}$.

III. Suppose $W = P$. In this case there is equilibrium when $\theta = 0$, no friction being then exerted; and besides this we have results which may be deduced from those in the first case. Here $\alpha = \epsilon$; if the coefficient of friction is less

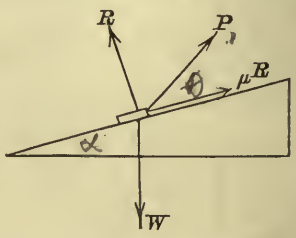
than unity there is no other position of equilibrium; if the coefficient of friction is greater than unity equilibrium will also subsist for all values of θ between $\pi - 2\epsilon$ and $\frac{\pi}{2}$.

Or the results for the third case may be deduced from those given in the second case, observing that $\alpha = \epsilon$.

Equilibrium of Machines with Friction.

178. *Inclined Plane.*

Let α be the inclination of the plane to the horizon. Suppose a force P_1 acting at an inclination θ to the plane and the body on the point of moving *down* the plane. Let R be the normal action of the plane, μR the friction which acts *up* the plane, W the weight of the body. Resolve the forces along and perpendicular to the plane; then, for equilibrium we have



$$P_1 \cos\theta + \mu R - W \sin\alpha = 0 \ldots\ldots\ldots\ldots\ldots (1),$$

$$R + P_1 \sin\theta - W \cos\alpha = 0 \ldots\ldots\ldots\ldots\ldots (2).$$

Substitute in (1) the value of R from (2); thus

$$P_1 = \frac{W \sin\alpha - \mu W \cos\alpha}{\cos\theta - \mu \sin\theta}.$$

Next, suppose P_2 a force acting at an inclination θ to the plane, such that the body is on the point of moving *up* the plane. Friction now acts *down* the plane, and we shall find

$$P_2 = \frac{W \sin\alpha + \mu W \cos\alpha}{\cos\theta + \mu \sin\theta}.$$

This result may be deduced from the former by changing the sign of μ.

There will be equilibrium if the body be acted on by a force P the magnitude of which lies between P_1 and P_2.

Suppose ϵ to be the *angle of friction*, so that $\tan \epsilon = \mu$,

then $$P_1 = \frac{W \sin \alpha - \tan \epsilon W \cos \alpha}{\cos \theta - \tan \epsilon \sin \theta} = \frac{W \sin (\alpha - \epsilon)}{\cos (\theta + \epsilon)}.$$

Similarly $$P_2 = \frac{W \sin (\alpha + \epsilon)}{\cos (\theta - \epsilon)}.$$

Suppose we require to know the least force which will suffice to prevent the body from moving down the plane. The expression for P_1 will be least when $\cos (\theta + \epsilon)$ is greatest, that is when $\theta + \epsilon = 0$, that is when $\theta = -\epsilon$; and then $P_1 = W \sin (\alpha - \epsilon)$.

Again, suppose we require to know the least force which will suffice to move the body up the plane. The expression for P_2 will be least when $\cos (\theta - \epsilon)$ is greatest, that is when $\theta = \epsilon$; and then $P_2 = W \sin (\alpha + \epsilon)$; a force which is just greater than this would move the body up the plane.

The following problem will illustrate the subject of the inclined plane with friction. A weight W is placed on a rough inclined plane, and is attached by a string to a point above the plane: determine the limiting positions of equilibrium.

Let α be the inclination of the plane to the horizon, β the inclination of the string to the plane, T the tension of the string, R the normal resistance of the plane. Since the body is constrained to remain at a constant distance from the fixed point, it must be situated on the circumference of a certain circle described on the plane; suppose θ the angular distance of the position of the body from the lowest point of the circumference. The forces which act on the body at right angles to the plane are $W \cos \alpha$, $T \sin \beta$, and R. Thus

$$R + T \sin \beta - W \cos \alpha = 0 \ldots\ldots\ldots\ldots\ldots (1).$$

The forces which act on the body in the plane are $W \sin \alpha$, $T \cos \beta$, and the friction μR. Resolve these forces along the radius and tangent at the point of the circumference at which the body rests. Thus

$$T \cos \beta - W \sin \alpha \cos \theta = 0 \ldots\ldots\ldots\ldots\ldots (2),$$

$$\mu R - W \sin \alpha \sin \theta = 0 \ldots\ldots\ldots\ldots\ldots (3).$$

From (1) and (3) we obtain

$$\mu T \sin\beta = \mu W \cos\alpha - W \sin\alpha \sin\theta;$$

hence by (2)

$$\mu \tan\beta = \frac{\mu \cos\alpha - \sin\alpha \sin\theta}{\sin\alpha \cos\theta}.$$

Therefore

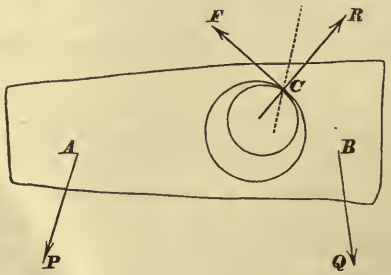
$$\mu = \frac{\sin\alpha \sin\theta}{\cos\alpha - \sin\alpha \tan\beta \cos\theta}$$

$$= \frac{\cot\beta \sin\theta}{\cot\alpha \cot\beta - \cos\theta}.$$

This result may then be developed in the manner already exemplified in Art. 177.

179. *Lever with Friction.*

Suppose a solid body pierced with a cylindrical hole through



which passes a solid fixed cylindrical axis. Let the outer circle in the figure represent a section of the cylindrical hole made by a plane perpendicular to its axis, and the inner circle the corresponding section of the solid axis. In the plane of this section, we suppose two forces P and Q to act on the solid body at the points A and B. Also at the point of contact C there is a normal force R and a tangential force F. These four forces keep the body in equilibrium.

Since R and F have a resultant passing through C, it follows, by Art. 58, that the resultant of P and Q must also pass

through C. Let γ be the angle between the directions of P and Q, and S the resultant of P and Q; then

$$S^2 = P^2 + Q^2 + 2PQ \cos \gamma.$$

Let the direction of S be represented by the dotted line making an angle θ with R. Then since F, R, and S are in equilibrium,

$$R = S \cos \theta \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$F = S \sin \theta \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

For the limiting position of equilibrium $F = \mu R$; therefore

$$\tan \theta = \mu \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

We may now find the relation between P and Q, by taking moments round the centre of the exterior circle; let r be the radius of this circle; a and b the distances of A and B from its centre; α and β the angles which the directions of P and Q make with these distances; then

$$Pa \sin \alpha + Fr = Qb \sin \beta;$$

or by (2) and (3),

$$Pa \sin \alpha + \frac{r\mu}{(1+\mu^2)^{\frac{1}{2}}} (P^2 + Q^2 + 2PQ \cos \gamma)^{\frac{1}{2}} = Qb \sin \beta \ldots (4).$$

If we suppose the friction to act in the opposite direction to that in the figure, we shall obtain

$$Pa \sin \alpha - \frac{r\mu}{(1+\mu^2)^{\frac{1}{2}}} (P^2 + Q^2 + 2PQ \cos \gamma)^{\frac{1}{2}} = Qb \sin \beta \ldots (5).$$

Equilibrium will not subsist unless P, Q, a, b, α, β, γ are so adjusted that (4) or (5) can be satisfied without giving to μ a value greater than its limit known by experiment.

The following form may be given to the limiting equation. Let s be the length of the perpendicular from the centre of the outer circle on the dotted line. Since F, R, and S are in equilibrium, we have by taking moments

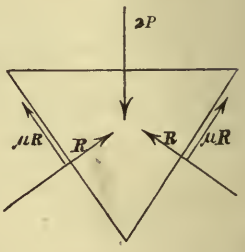
$$Fr = Ss;$$

therefore

$$\frac{r\mu}{(1+\mu^2)^{\frac{1}{2}}} = s.$$

180. *Wedge with Friction.* (See Art. 168.)

Suppose the wedge to be on the point of moving in the direction in which $2P$ urges it, and assume for simplicity that each face is similarly acted on by the obstacle. The forces which maintain the wedge in equilibrium are $2P$ perpendicular to the thick end, R perpendicular to each face, and μR along each face towards the thick end. Hence, resolving the forces parallel to the direction of $2P$,



$$P = R \sin \alpha + \mu R \cos \alpha \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Forces equal and opposite to R and μR act on the obstacle at each point of contact. If R' be the resultant of R and μR, we have

$$R' = R \sqrt{(1 + \mu^2)} \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Let S be the resolved part of R' along a direction making an angle i with that of R and i' with that of R' (see Art. 168); then

$$\begin{aligned} S &= R' \cos i' \\ &= R \cos i + \mu R \sin i \ldots\ldots\ldots\ldots\ldots\ldots (3). \end{aligned}$$

(1), (2), and (3) will give the ratio of P to R' and of P to S.

181. *Screw with Friction.* (See Arts. 169, 170.)

If the surfaces of the screw are rough it is kept in equilibrium by W, P, a system of forces perpendicular to the surface of the groove, and a system of forces arising from friction. Let R_r denote one of the forces perpendicular to the surface of the groove, μR_r the corresponding friction; then R_r makes an angle α with the axis of the cylinder on which the screw is raised, and μR_r an angle $\frac{1}{2}\pi - \alpha$ with the axis of the cylinder. Suppose W about to prevail over P; then resolving the forces parallel to the axis of the cylinder, and taking moments round it, we have

$$W - \Sigma R (\cos \alpha + \mu \sin \alpha) = 0,$$

$$Pa - \Sigma R (\sin \alpha - \mu \cos \alpha) b - 0.$$

Therefore
$$\frac{P}{W} = \frac{b\,(\sin\alpha - \mu\cos\alpha)}{a\,(\cos\alpha + \mu\sin\alpha)}$$
$$= \frac{b}{a}\frac{\tan\alpha - \mu}{1 + \mu\tan\alpha}$$
$$= \frac{b}{a}\tan(\alpha - \epsilon),$$
if
$$\mu = \tan\epsilon.$$

If we suppose P about to prevail over W, we shall find similarly
$$\frac{P}{W} = \frac{b}{a}\tan(\alpha + \epsilon).$$

[The above investigation is applicable to a "square-threaded" screw, that is, a screw in which a section of the thread by a plane through the axis is rectangular. When the section is wedge-shaped, as in an ordinary screw-nail, R and μR are greater for the same value of W. See No. 12 of the following examples.]

EXAMPLES.

1. A rectangular prism, whose breadth is 2·8$\dot{3}$ feet and thickness less than 2 inches, is laid with its axis horizontal, and with its smaller face on an inclined plane where the coefficient of friction is $\frac{1}{17}$. Shew that if the inclination of the plane is gradually increased, the prism will roll before it will slide.

2. If the roughness of a plane which is inclined to the horizon at a known angle be such that a body will just rest supported on it, find the least force requisite to draw the body up.

Results. Let α be the inclination of the plane, W the weight of the body; then the force must be just greater than $W \sin 2\alpha$, and act at an inclination α to the plane.

3. Two rough bodies rest on an inclined plane, and are connected by a string which is parallel to the plane; if the coefficient of friction be not the same for both, find the

greatest inclination of the plane which is consistent with equilibrium.

Result. $\tan\theta = \dfrac{\mu_1 W_1 + \mu_2 W_2}{W_1 + W_2}$.

4. A rectangular table stands on a rough inclined plane, and has two sides horizontal: if the coefficient of friction of the lowest feet be μ and that of the others be μ', find the inclination of the plane when the table is on the point of sliding.

5. Two unequal weights on a rough inclined plane are connected by a string which passes through a fixed pully in the plane: find the greatest inclination of the plane consistent with the equilibrium of the weights.

Result. $\tan\alpha = \dfrac{\mu(W_1 + W_2)}{W_1 - W_2}$.

6. A heavy uniform rod whose length is $2a$ is supported by resting on a rough peg, a string of length l being attached to one end of the rod and fastened to a given point in the same horizontal plane with the peg. If when the rod is on the point of sliding the string is perpendicular to it the coefficient of friction is $\dfrac{l}{a}$.

7. Two weights P, Q of similar material rest on a rough double inclined plane, and are connected by a fine string passing over the common vertex: Q is on the point of motion down the plane, shew that the weight which may be added to P without producing motion is

$$\frac{P \sin 2\phi \sin(\alpha + \beta)}{\sin(\beta - \phi)\sin(\alpha - \phi)},$$

α, β being the angles of inclination of the planes and $\tan\phi$ the coefficient of friction.

8. A weight P is attached to a point in the circumference of a rough circular ring whose weight is W: shew that the ring will hang on a horizontal rod in a plane perpendicular to

it with any point of the ring in contact with the rod, if the coefficient of friction be not less than

$$\frac{1}{\sqrt{(n^2+2n)}}, \text{ where } n=\frac{W}{P}.$$

9. Two equal heavy rings are moveable on a horizontal rough rod; a string of given length which passes through them has both ends attached to a given weight: find the greatest possible distance between the rings.

10. Three equal hemispheres, having their bases downwards, are placed in contact with each other upon a horizontal table; if a smooth sphere of the same substance and equal radius be placed upon them, shew that there will be equilibrium or not, according as the coefficient of friction between the hemispheres and the table is greater or less than $\frac{1}{3}\sqrt{2}$.

11. A uniform rod rests wholly within a rough hemispherical bowl in a vertical plane through its centre, prove that the limiting position of equilibrium will be given by the equation

$$\tan\theta=\frac{\sin 2\epsilon}{2\cos(\beta+\epsilon)\cos(\beta-\epsilon)},$$

θ being the inclination of the rod to the horizon, 2β the angle it subtends at the centre, and $\tan\epsilon$ the coefficient of friction.

12. A thin rod rests in a horizontal position between two rough planes equally inclined to the horizon, and whose inclination to each other is 2α: if μ be the coefficient of friction, then the greatest possible inclination of the line of intersection of the planes to the horizon is $\tan^{-1}\frac{\mu}{\sin\alpha}$.

13. A surface is formed by the revolution of an equilateral hyperbola about one of its asymptotes which is vertical; shew that a particle will rest upon it, supposing it rough, anywhere beyond the intersection of the surface with a certain circular cylinder.

14. A heavy particle under the action of gravity will rest on a rough paraboloid $\frac{x^2}{\alpha} + \frac{y^2}{\beta} = 2z$, if it be placed on the surface at any point above the curve of intersection of the surface with the cylinder $\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = \mu^2$; the axis of the paraboloid being vertical, its vertex upwards, and μ the coefficient of friction.

15. A rough elliptic pully of weight W can turn freely about one extremity of its major axis, and two weights, P, Q, are suspended by a string which passes over the pully; when in equilibrium its plane is vertical, and its axis inclined at 60^0 to the horizon; prove that the excentricity of the ellipse is equal to

$$\frac{\sqrt{\{(3Q + W - P)(Q - W - 3P)\}}}{(Q - P)\sqrt{3}}.$$

16. A heavy hemisphere rests with its convex surface on a rough inclined plane. Find the greatest possible inclination of the plane.

17. One end A of a heavy rod ABC rests against a rough vertical plane; and a point B of the rod is connected with a point in the plane by a string, the length of which is equal to AB: determine the position of equilibrium of the rod, and shew how the direction in which the friction acts depends upon the position of B.

18. Three equal balls, placed in contact on a horizontal plane, support a fourth ball. Determine the least values of the coefficients of friction of the balls with each other and with the plane, that the equilibrium may be possible.

Results. Let W be the weight of each of the three lower balls, W' the weight of the upper, ϕ the angle which the straight line joining the centre of the upper ball with the centre of one of the lower balls makes with the vertical; then the coefficient of the friction between the balls is $\tan \frac{\phi}{2}$, and

the coefficient of the friction between the balls and the plane is $\frac{W'}{W'+3W}\tan\frac{\phi}{2}$. If all four balls are equal we have $\sin\phi=\frac{1}{\sqrt{3}}$, so that $\tan\frac{\phi}{2}=\sqrt{3}-\sqrt{2}$.

19. Determine the curve on the rough surface of an ellipsoid, at every point of which a particle acted on by three equal forces whose directions are parallel to the axes of the ellipsoid, will rest in a limiting position of equilibrium.

20. $BCDE$ is a square board; a string is fixed to a point A in a rough wall and to the corner B of the board. Shew that the board will rest with its plane perpendicular to the wall, and its side CD resting against it, if AC be not greater than μBC.

21. A rectangular parallelepiped of given dimensions is placed with one face in contact with a rough inclined plane; determine the limits of its position in order that equilibrium may exist.

22. A board, moveable about a horizontal straight line in its own plane, is supported by resting on a rough sphere which lies on a horizontal table: find the greatest inclination at which the board can rest.

Result. Let θ be the inclination of the board to the horizon; then $\tan\frac{\theta}{2}=\mu$, where μ is the greater of the two coefficients of friction.

23. A string PCB passes over a smooth pully C, and has a given weight P attached to one extremity, while the other extremity B is attached to one end of a heavy uniform beam AB at B. The other end A of the beam rests upon a rough horizontal plane; determine the position of the beam when in equilibrium.

24. A hemisphere is supported by friction with its curved surface in contact with a horizontal and vertical plane: find the limiting position of equilibrium.

Result. If θ be the inclination of the plane base to the horizon, $\sin\theta = \dfrac{8\mu(1+\mu)}{3(1+\mu^2)}$.

25. When a person tries to pull out a two-handled drawer by pulling one of its handles in a direction perpendicular to its front, find the condition under which the drawer will stick fast.

26. Determine the condition under which a weight can be supported on a rough vertical screw without the action of any force; for example, if the coefficient of friction be $\frac{1}{2}$, find the least number of turns which may be given to a thread on a cylinder 2 feet long and 6 inches in circumference. *Result.* Eight.

27. Two uniform beams of the same length and material placed in a vertical plane, are in a state of rest bordering on motion under the following circumstances: their upper ends are connected by a smooth hinge, about which they can move freely; their other ends rest on a rough horizontal plane, and the beams are perpendicular to each other: find the coefficient of friction between the beams and the horizontal plane. *Result.* $\mu = \frac{1}{2}$.

28. A straight uniform beam is placed upon two rough planes whose inclinations to the horizon are α and α', and the coefficients of friction $\tan\lambda$ and $\tan\lambda'$: shew that if θ be the limiting value of the angle of inclination of the beam to the horizon at which it will rest, W its weight, and R, R' the pressures upon the planes

$$2\tan\theta = \cot(\alpha'+\lambda') - \cot(\alpha-\lambda),$$

$$\frac{R}{\cos\lambda\sin(\alpha'+\lambda')} = \frac{R'}{\cos\lambda'\sin(\alpha-\lambda)} = \frac{W}{\sin(\alpha-\lambda+\alpha'+\lambda')}.$$

29. A heavy right cylinder is placed with its base on a rough horizontal plane, and is capable of motion round its axis: find the least couple in a horizontal plane which will move it.

30. Two weights of different material are laid on an inclined plane connected by a string extended to its full length, inclined at an angle θ to the line of intersection of the inclined plane with the horizon; if the lower weight be on the point of motion find the magnitude and direction of the force of friction on the upper weight.

31. A carriage stands upon four equal wheels; given the coefficient of friction between the axles and the wheels find the greatest slope on which it can remain at rest neglecting the weight of the wheels.

CHAPTER XI.

FLEXIBLE INEXTENSIBLE STRINGS.

182. A STRING is said to be *perfectly flexible* when any force, however small, which is applied otherwise than along the direction of the string, will change its form. For shortness, we use the word *flexible* as equivalent to *perfectly flexible*. Sometimes the word *chain* is used as synonymous with *string*.

If a flexible string be kept in equilibrium by two forces, one at each end, we assume as self-evident that those forces must be equal and act in opposite directions, so that the string assumes the form of a straight line in the direction of the forces. In this case the *tension* of the string is measured by the force applied at one end.

Let ABC represent a string kept in equilibrium by a force T at one end A and an equal force T at the other end C acting in opposite directions along the line AC. Since any portion AB of the string is in equilibrium it follows that a force T must act on AB at B from B towards C in order to balance the force acting at A; and similarly, a force T must act on BC from B towards A in order that BC may be in equilibrium. This result is expressed by saying that *the tension of the string is the same throughout.*

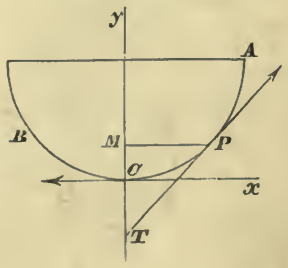
Unless the contrary be expressed, a string is supposed *inextensible* and the boundary of a transverse section of it is supposed to be a curve every chord of which is indefinitely small.

183. When a flexible string is acted on by other forces besides one at each end it may in equilibrium assume a *curvilinear* form. If at any point of the curve we suppose

a section made by a plane perpendicular to the tangent, the mutual action of the portions on opposite sides of this plane must be in the direction of the tangent, or else equilibrium would not hold, since the string is perfectly flexible.

184. *A heavy string of uniform density and thickness is suspended from two given points; required to find the equation to the curve in which the string hangs when it is in equilibrium.*

Let A, B be the fixed points to which the ends are attached; the string will rest in a *vertical plane* passing through A and B, because there is no reason why it should deviate to one side rather than the other of this vertical plane. Let ACB be the form it assumes, C being the lowest point; take this as the origin of co-ordinates; let P be any point in the curve; CM, which is vertical, $= y$; MP, which is horizontal, $= x$; $CP = s$.



The equilibrium of any portion CP will not be disturbed if we suppose it to become rigid. Let c and t be the lengths of portions of the string of which the weights equal the tensions at C and P. Then CP is a rigid body acted on by three forces which are proportional to c, s, and t, and act respectively, horizontally, vertically, and along the tangent at P. Draw PT the tangent at P meeting the axis of y in T; then the forces holding CP in equilibrium have their directions parallel to the sides of the triangle PMT, and therefore bear the same proportion one to another that these sides do (see Art. 19); therefore

$$\frac{PM}{MT} = \frac{\text{tension at lowest point}}{\text{weight of the portion } CP}, \quad \text{or} \quad \frac{dx}{dy} = \frac{c}{s},$$

therefore
$$\frac{dy}{dx} = \frac{s}{c} \text{ and } \frac{dy}{ds} = \frac{s}{\sqrt{(c^2 + s^2)}};$$

therefore
$$y + c = \sqrt{(c^2 + s^2)} \ldots\ldots\ldots\ldots\ldots\ldots (1);$$

the constant added being such that $y = 0$ when $s = 0$; therefore

$$s^2 = y^2 + 2yc \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Also $$\frac{dx}{dy} = \frac{c}{s} = \frac{c}{\sqrt{(y^2 + 2yc)}},$$

therefore $$x = c \log \frac{y + c + \sqrt{(y^2 + 2yc)}}{c} \quad \ldots\ldots\ldots\ldots\ldots\ldots (3),$$

the constant being chosen so that x and y vanish together. The last equation gives

$$ce^{\frac{x}{c}} = y + c + \sqrt{(y^2 + 2yc)}.$$

Transpose and square; thus

$$c^2 e^{\frac{2x}{c}} - 2(y + c)\, ce^{\frac{x}{c}} + c^2 = 0;$$

therefore $$y + c = \tfrac{1}{2}c\,(e^{\frac{x}{c}} + e^{-\frac{x}{c}}) \quad \ldots\ldots\ldots\ldots\ldots\ldots (4).$$

Also $$s = \sqrt{\{(y + c)^2 - c^2\}} \text{ by (2)}$$

$$= \tfrac{1}{2}c\,(e^{\frac{x}{c}} - e^{-\frac{x}{c}}) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5).$$

Any one of these five equations may be taken as the equation to the curve. If in equation (4) we write y' for $y + c$, which amounts to moving the origin to a point vertically below the lowest point of the curve at a distance c from it, we have

$$y' = \tfrac{1}{2}c\,(e^{\frac{x}{c}} + e^{-\frac{x}{c}}).$$

When the string is uniform in density and thickness, as in the present instance, the curve is called the *common catenary*.

185. *To find the tension of the string at any point.*

Let the tension at P be equal to the weight of a length t of the string; then, as shewn in the last Article,

$$\frac{\text{tension at } P}{\text{weight of } CP} = \frac{PT}{TM}, \text{ therefore } \frac{t}{s} = \frac{ds}{dy}.$$

But $s^2 = y^2 + 2yc$ by equation (2) of Art. 184, therefore

$$t = y + c = y'.$$

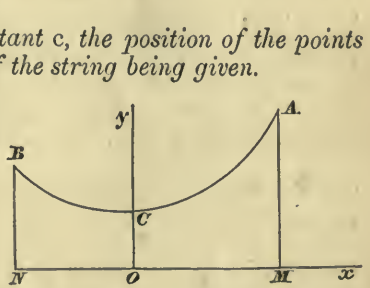
This shews that the tension at any point is the weight of a portion of string whose length is the ordinate at that point, the origin being at a distance c below the lowest point.

Hence, if a uniform string hang freely over any two points, the extremities of the string will lie in the same horizontal line when the string is in equilibrium.

186. *To determine the constant* c, *the position of the points of suspension and the length of the string being given.*

Let A and B be the fixed extremities, C the lowest point of the curve.

$$OC = c, \quad OM = a,$$
$$ON = a', \quad MA = b,$$
$$NB = b', \quad CA = l, \quad CB = l'.$$



Also let

$$\left.\begin{aligned} a + a' &= h \\ b - b' &= k \\ l + l' &= \lambda \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1);$$

then h, k, λ are known quantities, since the length of the string and the positions of its ends are given. From Art. 184

$$\left.\begin{aligned} b &= \tfrac{1}{2}c\,(e^{\frac{a}{c}} + e^{-\frac{a}{c}}) \\ b' &= \tfrac{1}{2}c\,(e^{\frac{a'}{c}} + e^{-\frac{a'}{c}}) \\ l &= \tfrac{1}{2}c\,(e^{\frac{a}{c}} - e^{-\frac{a}{c}}) \\ l' &= \tfrac{1}{2}c\,(e^{\frac{a'}{c}} - e^{-\frac{a'}{c}}) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Equations (1) and (2) are theoretically sufficient to enable us to eliminate a, a', b, b', l, and l' and to determine c. We may deduce from them

$$\lambda = \tfrac{1}{2}c\,(e^{\frac{a}{c}} - e^{-\frac{a}{c}} + e^{\frac{a'}{c}} - e^{-\frac{a'}{c}}),$$

$$k = \tfrac{1}{2}c\,(e^{\frac{a}{c}} + e^{-\frac{a}{c}} - e^{\frac{a'}{c}} - e^{-\frac{a'}{c}});$$

therefore $$\lambda + k = c\,(e^{\frac{a}{c}} - e^{-\frac{a'}{c}}),$$

$$\lambda - k = c\,(e^{\frac{a'}{c}} - e^{-\frac{a}{c}});$$

therefore $$\lambda^2 - k^2 = c^2\,(e^{\frac{a+a'}{c}} + e^{-\frac{a+a'}{c}} - 2)$$

$$= c^2\,(e^{\frac{h}{c}} + e^{-\frac{h}{c}} - 2);$$

therefore $$\sqrt{(\lambda^2 - k^2)} = c\,(e^{\frac{h}{2c}} - e^{-\frac{h}{2c}}) \quad \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

This is the equation from which c is to be found, but on account of its transcendental form it can only be solved by approximation. If the exponents of e are small, we may expand by the exponential theorem and thus obtain the approximate value of c. In order that the exponents may be small, c must be large compared with h; since $\frac{dy}{ds} = \frac{s}{\sqrt{(c^2 + s^2)}}$ by Art. 184 it follows that when c is large, compared with the length of the string, $\frac{dy}{ds}$ is small, and therefore the curve does not deviate much from a horizontal straight line. Hence, when the two points of support are nearly in a horizontal line and the distance between them nearly equal to the given length of the string, we may conclude that $\frac{h}{c}$ will be small. In this case, we have from (3)

$$\sqrt{(\lambda^2 - k^2)} = 2c\left\{\frac{h}{2c} + \frac{1}{\lfloor 3}\left(\frac{h}{2c}\right)^3 + \frac{1}{\lfloor 5}\left(\frac{h}{2c}\right)^5 + \ldots\right\} \quad \ldots\ldots (4);$$

therefore $$\sqrt{(\lambda^2 - k^2)} = h + \frac{h^3}{24c^2} \text{ approximately.}$$

The right-hand member of (4) continually decreases as c^2 increases; so that there cannot be more than one value of c^2 which satisfies the equation.

187. *To find the equations of equilibrium when a flexible string is acted on by any forces.*

Let x, y, z be the co-ordinates of a point P of the string; let s denote the length of the curve BP measured from some

fixed point B up to P, and δs the length of the arc PQ between P and an adjacent point Q. Let κ be the area of a section of the string at P, and ρ the density at P; let T be the tension of the string at P; then $T\frac{dx}{ds}$, $T\frac{dy}{ds}$, and $T\frac{dz}{ds}$ are the resolved parts of T parallel to the co-ordinate axes; and the resolved parts of the tension at Q parallel to the axes will be, by Taylor's Theorem,

$$T\frac{dx}{ds}+\frac{d}{ds}\left(T\frac{dx}{ds}\right)\delta s+\text{terms in }(\delta s)^2,\ \&c.$$

$$T\frac{dy}{ds}+\frac{d}{ds}\left(T\frac{dy}{ds}\right)\delta s+\ldots\ldots\ldots\ldots,$$

$$T\frac{dz}{ds}+\frac{d}{ds}\left(T\frac{dz}{ds}\right)\delta s+\ldots\ldots\ldots\ldots$$

Let $X\rho\kappa\delta s$, $Y\rho\kappa\delta s$, $Z\rho\kappa\delta s$ be the external forces which act on the element PQ parallel to the axes. The equilibrium of the element will not be disturbed by supposing it to become rigid; hence, by Art. 27, the sum of the forces parallel to the axis of x must vanish; thus

$$T\frac{dx}{ds}+\frac{d}{ds}\left(T\frac{dx}{ds}\right)\delta s+\ldots+X\rho\kappa\delta s-T\frac{dx}{ds}=0,$$

or
$$\frac{d}{ds}\left(T\frac{dx}{ds}\right)+X\rho\kappa=0 \text{ ultimately.}$$

Similarly $\frac{d}{ds}\left(T\frac{dy}{ds}\right)+Y\rho\kappa=0$, and $\frac{d}{ds}\left(T\frac{dz}{ds}\right)+Z\rho\kappa=0$.

The product $\kappa\rho$ may be conveniently replaced by m, so that if m be constant ml represents the mass of a length l of the string, and therefore m the mass of a unit of length of the string. If m be not constant, conceive a string having its length equal to the unit of length and its section and density throughout the same as those of the given string at

the point (x, y, z), and then m will be the mass of such supposed string.

The element δs of the string, the equilibrium of which we have considered, becomes more nearly a *particle* the more we diminish δs; hence it is sufficient to consider the three equations of Art. 27 instead of the six equations of Art. 73.

The three equations which we have found are theoretically sufficient for determining T, y, and z as functions of x, remembering that $\frac{ds}{dx} = \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2 + \left(\frac{dz}{dx}\right)^2\right\}}$; and when we know the values of y and z in terms of x, we know the equations to the curve which the string forms.

188. The equations for the equilibrium of a flexible string may be written thus:

$$\left.\begin{aligned} T\frac{d^2x}{ds^2} + \frac{dT}{ds}\frac{dx}{ds} + mX = 0 \\ T\frac{d^2y}{ds^2} + \frac{dT}{ds}\frac{dy}{ds} + mY = 0 \\ T\frac{d^2z}{ds^2} + \frac{dT}{ds}\frac{dz}{ds} + mZ = 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Multiply these equations by $\frac{dx}{ds}$, $\frac{dy}{ds}$, and $\frac{dz}{ds}$ respectively and add; then, since

$$\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2 = 1,$$

and

$$\frac{dx}{ds}\frac{d^2x}{ds^2} + \frac{dy}{ds}\frac{d^2y}{ds^2} + \frac{dz}{ds}\frac{d^2z}{ds^2} = 0,$$

we have

$$\frac{dT}{ds} + m\left(X\frac{dx}{ds} + Y\frac{dy}{ds} + Z\frac{dz}{ds}\right) = 0 \ldots\ldots (2);$$

therefore

$$T + \int m\left(X\frac{dx}{ds} + Y\frac{dy}{ds} + Z\frac{dz}{ds}\right) ds = \text{constant} \ldots (3).$$

If the forces are such that $m(Xdx + Ydy + Zdz)$ is the immediate differential of some function of x, y, z, as $f(x, y, z)$, then

$$T + f(x, y, z) = \text{constant}.$$

If the forces are such that their resultant at every point of the curve is perpendicular to the tangent at that point, we have

$$X\frac{dx}{ds} + Y\frac{dy}{ds} + Z\frac{dz}{ds} = 0,$$

therefore, by (3), T is constant.

In the equations (1) transpose the terms mX, mY, mZ to the right-hand side, then square and add; thus

$$T^2\left\{\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2\right\} + \left(\frac{dT}{ds}\right)^2 = m^2(X^2 + Y^2 + Z^2).$$

Hence if ρ be the radius of absolute curvature of the curve formed by the string, and $Fm\delta s$ the resultant external force on the element δs, so that $F^2 = X^2 + Y^2 + Z^2$,

$$\left(\frac{T}{\rho}\right)^2 + \left(\frac{dT}{ds}\right)^2 = m^2F^2 \ldots\ldots\ldots\ldots\ldots\ldots (4).$$

If T be constant $\frac{dT}{ds} = 0$; hence in this case mF varies as $\frac{1}{\rho}$.

From the equations of equilibrium in Art. 187, we deduce by integration,

$$T\frac{dx}{ds} = -\int mXds,$$

$$T\frac{dy}{ds} = -\int mYds,$$

$$T\frac{dz}{ds} = -\int mZds.$$

Square and add; then

$$T^2 = \{\int mXds\}^2 + \{\int mYds\}^2 + \{\int mZds\}^2 \ldots\ldots\ldots (5).$$

The constants that enter when we integrate the differential equations of equilibrium must be determined from the special

circumstances of each particular problem. Thus the coordinates of fixed points to which the ends of the string are attached may be given, and the length of the string. Or, besides the forces represented by $mX\delta s$, $mY\delta s$, and $mZ\delta s$ acting on each element, given forces F_1 and F_2 may act at the extremities of the string; in this case if T_1 and T_2 denote the values of T at the two extremities of the string, we must have T_1 equal in magnitude to F_1 and opposite to it in direction, and similarly for T_2 and F_2.

189. From equations (1) of Art. 188, eliminate T and $\frac{dT}{ds}$; then we have

$$X\left(\frac{d^2y}{ds^2}\frac{dz}{ds}-\frac{d^2z}{ds^2}\frac{dy}{ds}\right)+Y\left(\frac{d^2z}{ds^2}\frac{dx}{ds}-\frac{d^2x}{ds^2}\frac{dz}{ds}\right)$$
$$+Z\left(\frac{d^2x}{ds^2}\frac{dy}{ds}-\frac{d^2y}{ds^2}\frac{dx}{ds}\right)=0;$$

this shews that the resultant external force which acts on an element δs of the string lies in the osculating plane at the point (x, y, z).

190. The general equations of equilibrium become, when all the forces are in one plane, namely, that of (x, y),

$$\frac{d}{ds}\left(T\frac{dx}{as}\right)+mX=0, \quad \frac{d}{ds}\left(T\frac{dy}{ds}\right)+mY=0 \ldots\ldots(1).$$

Suppose $X=0$, so that the external force is parallel to the axis of y; the first equation gives

$$T\frac{dx}{ds}=\text{a constant, } C \text{ say,}$$

therefore

$$T=\frac{C}{\frac{dx}{ds}} \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Hence the second equation becomes

$$C\frac{d}{ds}\left(\frac{dy}{dx}\right)+mY=0,$$

or

$$C\frac{d^2y}{dx^2}\frac{dx}{ds}+mY=0 \ldots\ldots\ldots\ldots(3).$$

For example; required the form of the curve when its weight is the only force acting on it, and the area of the section at any point is proportional to the tension at that point. Here Y is constant and may be denoted by $-g$, the axis of y being vertically upwards. And T varies as m, so that $T = \lambda m$ where λ is a constant. Thus from (2) and (3) we obtain

$$\frac{d^2y}{dx^2}\left(\frac{dx}{ds}\right)^2 = \frac{g}{\lambda}.$$

Put a for $\frac{\lambda}{g}$; thus $\frac{d^2y}{dx^2}\left(\frac{dx}{ds}\right)^2 = \frac{1}{a}$,

therefore
$$\frac{\frac{d^2y}{dx^2}}{1+\left(\frac{dy}{dx}\right)^2} = \frac{1}{a},$$

therefore
$$\tan^{-1}\frac{dy}{dx} = \frac{x}{a} + \text{constant}.$$

The constant vanishes if we suppose the origin at the lowest point of the curve; therefore

$$\frac{dy}{dx} = \tan\frac{x}{a};$$

therefore
$$\frac{y}{a} = -\log\cos\frac{x}{a} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

Since in this case the area of the section at any point is proportional to the tension at that point, the curve determined by (4) is called the *Catenary of equal strength.*

Since $T = \lambda m = mag$, we have the following result: the tension at any point is equal to the weight of a length a of a uniform string of the same area and density as the string actually has at the point considered.

Again, to find the curve assumed under the action of gravity by a string in which the mass is so distributed that portions which have equal horizontal projections have equal weights.

Since $m\delta s$ is constant when δx is constant, we have $m\frac{ds}{dx} = \text{constant} = k$, suppose.

Hence equation (3), putting $-g$ for Y, becomes

$$C\frac{d^2y}{dx^2} - kg = 0, \quad \frac{d^2y}{dx^2} = \frac{kg}{C}$$

$$y = \frac{kg}{2C}x^2 + Ax + B,$$

denoting a parabola with its axis vertical. This investigation is applicable to the supporting chains of a suspension bridge.

191. The equations (1) of the preceding Article may be written

$$T\frac{d^2x}{ds^2} + \frac{dT}{ds}\frac{dx}{ds} + mX = 0 \quad \text{.....(1).}$$

$$T\frac{d^2y}{ds^2} + \frac{dT}{ds}\frac{dy}{ds} + mY = 0 \quad \text{.....(2).}$$

Multiply (1) by $\frac{dy}{ds}$ and (2) by $\frac{dx}{ds}$ and subtract; thus

$$T\left(\frac{d^2x}{ds^2}\frac{dy}{ds} - \frac{d^2y}{ds^2}\frac{dx}{ds}\right) + m\left(X\frac{dy}{ds} - Y\frac{dx}{ds}\right) = 0,$$

from which, since $\frac{dx}{ds}\frac{d^2x}{ds^2} + \frac{dy}{ds}\frac{d^2y}{ds^2} = 0$, we find

$$T = \frac{m\dfrac{dx}{ds}}{\dfrac{d^2y}{ds^2}}\left(X\frac{dy}{ds} - Y\frac{dx}{ds}\right) \text{.....(3).}$$

Again, multiply (1) by $\frac{dx}{ds}$ and (2) by $\frac{dy}{ds}$ and add; then

$$\frac{dT}{ds} + m\left(X\frac{dx}{ds} + Y\frac{dy}{ds}\right) = 0 \text{.....(4).}$$

From (3) and (4) by eliminating T, we deduce

$$m\left(X\frac{dx}{ds} + Y\frac{dy}{ds}\right) + \frac{d}{ds}\left\{\frac{m\dfrac{dx}{ds}}{\dfrac{d^2y}{ds^2}}\left(X\frac{dy}{ds} - Y\frac{dx}{ds}\right)\right\} = 0,$$

which is the general equation to the curve when given forces act in one plane.

192. In Art. 188 we have found the equations

$$\frac{dT}{ds} + m\left(X\frac{dx}{ds} + Y\frac{dy}{ds} + Z\frac{dz}{ds}\right) = 0 \quad \ldots\ldots(1),$$

$$\left(\frac{T}{\rho}\right)^2 + \left(\frac{dT}{ds}\right)^2 = m^2F^2 \quad \ldots\ldots(2).$$

Let ϕ be the angle which the resultant force $mF\delta s$ makes with the tangent at the point (x, y, z); then

$$F\cos\phi = X\frac{dx}{ds} + Y\frac{dy}{ds} + Z\frac{dz}{ds};$$

therefore by (1),

$$\frac{dT}{ds} = -mF\cos\phi \quad \ldots\ldots(3),$$

therefore by (2),

$$\left(\frac{T}{\rho}\right)^2 = m^2F^2\sin^2\phi \quad \ldots\ldots(4).$$

If the force be such that its direction always passes through a fixed point, the whole string will lie in a plane passing through its ends and through the fixed point, for there is no reason why it should lie on one side rather than the other of this plane. Let r be the distance of the point (x, y, z) of the curve from the fixed point, p the perpendicular from the fixed point on the tangent at (x, y, z); then (3) and (4) may be written

$$\frac{dT}{ds} = -mF\frac{dr}{ds} \quad \ldots\ldots(5),$$

$$\frac{T}{\rho} = mF\frac{p}{r} \quad \ldots\ldots(6).$$

Hence $$\frac{1}{T}\frac{dT}{ds} = -\frac{r}{\rho p}\frac{dr}{ds} = -\frac{1}{p}\frac{dp}{ds};$$

therefore $$\log T = \text{constant} - \log p,$$

or $$Tp = C.$$

The equation $Tp = C$ may also be obtained simply thus: suppose a finite portion of the string to become rigid; this portion is acted on by the tensions at its two ends and by *other forces which all pass through a fixed point;* take moments round this fixed point; hence the product of the tension into the perpendicular from the fixed point on the tangent must have the same value at the two ends of the finite portion of the string. Thus $Tp =$ constant.

From (5) we have

$$mF = -\frac{dT}{dr} = -\frac{d}{dr}\frac{C}{p} = \frac{C}{p^2}\frac{dp}{dr} \quad \ldots\ldots\ldots\ldots\ldots\ldots (7);$$

hence mF (which may be called the central force per unit length of the string) can be found when the curve formed by the string is given; and conversely when mF is given in terms of r the equations to the possible curves can be found[1].

If mF (regarded as positive for repulsion) varies as the nth power of the distance, we have

$$\frac{d}{dr}\frac{C}{p} = -\mu r^n, \qquad \frac{C}{p} = C' - \mu\frac{r^{n+1}}{n+1},$$

except when $n = -1$, in which case $\dfrac{C}{p} = C' - \mu \log r$.

The results for attraction will be

$$\frac{C}{p} = C' + \mu\frac{r^{n+1}}{n+1}, \qquad \frac{C}{p} = C' + \mu \log r.$$

For the parabola referred to its focus we have $p^2 = ar$; hence by (7), mF varies as $r^{-\frac{5}{2}}$.

[1] It is proved in treatises on the Dynamics of a Particle that the central attractive force under which a particle can describe a given orbit is $\dfrac{h^2}{p^3}\dfrac{dp}{dr}$, where h denotes a constant. Calling this central force P, equation (7) shews that mF has a constant ratio to Pp when the curve and centre of force are given. The directions of F and P are opposite.

When the centre is very distant p varies as $\sin\alpha$, where α denotes the inclination of the curve to the direction of the force; hence mF varies as $P \sin\alpha$. When P is constant the curve is a parabola; hence in a parabolic chain under the action of gravity m varies as $\sin\alpha$.

When mF is constant, the curve is the common catenary.

Hence for a free particle to move in a catenary under a force parallel to the axis, the force must vary as $\operatorname{cosec}\alpha$.

For the equiangular spiral referred to its pole $p \propto r$, hence $mF \propto r^{-2}$.

For the rectangular hyperbola referred to its centre, $pr = a^2$; hence mF is constant.

For the cardioid referred to its cusp, $r^3 = 2ap^2$, hence $mF \propto r^{-\frac{5}{2}}$.

For the lemniscate of Bernouilli referred to its centre, $r^3 = a^2 p$, hence $mF \propto r^{-4}$.

When the equation of the curve is given or required in terms of r and θ we have only to write for $\frac{1}{p}$ its value

$$\left\{\frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2\right\}^{\frac{1}{2}}.$$

It is obvious from the general equations of Art. 187 that if all the external forces acting on a string are increased or diminished in any constant ratio, this change will not alter the form of the string, but will merely produce a proportionate change in the tensions at all its points.

193. Equations (3) and (4) of the preceding article, when freed from the ambiguity of sign which attaches to one of them, may be stated in the following words:

The tangential component of the applied force, reckoned positive when it is in the direction in which s increases *is equal to minus* $\frac{dT}{ds}$.

The normal component of the applied force is directed towards the convex side of the curve, in the osculating plane, and *is equal to* $\frac{T}{\rho}$.

These may be regarded as the fundamental equations of the theory of flexible strings, and can be directly proved as follows:

Resolving along the tangent at the beginning of the element δs, for the equilibrium of the element $m\delta s$, we have

$$(T + \delta T)\cos\frac{\delta s}{\rho} + Fm\delta s \cos\phi - T = 0.$$

But $\cos\frac{\delta s}{\rho}$ differs from unity by a small quantity of the second order. Hence to the first order of small quantities,

$$\delta T + Fm\delta s \cos\phi = 0, \quad \frac{dT}{ds} + Fm\cos\phi = 0 \ldots\ldots\ldots(1).$$

Again, resolving along the normal in the osculating plane,

$$(T + dT)\sin\frac{\delta s}{\rho} - Fm\delta s \sin\phi = 0.$$

But to the first order of small quantities, $\sin\frac{\delta s}{\rho} = \frac{\delta s}{\rho}$;

hence $$T\frac{\delta s}{\rho} - Fm\delta s \sin\phi = 0, \quad \frac{T}{\rho} - Fm\sin\phi = 0 \ldots\ldots(2).$$

Equation (1) gives for the increase of tension in travelling a small distance along the curve, $\delta T = -Fm\cos\phi\,\delta s$. But $\cos\phi\,\delta s$ is the projection of δs on the direction of F; hence when the direction of F is constant and m is also constant, the difference of the tensions at two points is the product of Fm into the projection of the intervening arc on the direction of F, the tension *diminishing* when we travel *with* F. For example, when gravity is the force, the difference of the tensions at two points of a uniform chain is the product of mg into the difference of their heights, the upper point having the greater tension. In a non-uniform chain, the difference of the tensions at two points is given by the following rule:

Suppose a vertical chain whose two ends are at the same levels as the given points, and let the value of m be the same in the two chains for points at the same level; the weight of this vertical chain will be equal to the required difference of tensions.

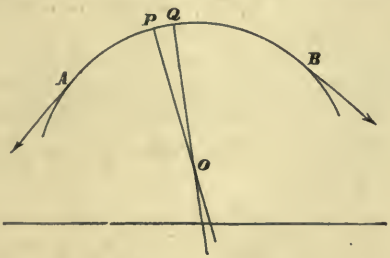
If a particle, of mass m, moves over the arc δs, the work done upon it is $Fm\cos\phi\,\delta s$, and this from above is equal to $-\delta T$. Hence if m is uniform, the diminution of tension in travelling over any portion of the string is equal to the work which would be done upon a particle of mass m in moving over the same portion. In ordinary cases the amount of this work depends only on the positions of the two ends of the

portion, and is independent of the length of string by which they are joined.

194. *A string is stretched over a smooth plane curve; to find the tension at any point and the pressure on the curve.*

First suppose the weight of the string neglected.

Let $APQB$ be the string, A and B being the points where



it leaves the curve. Let P, Q be adjacent points in the string; let the normals to the curve at P and Q meet at O; let θ be the angle which PO makes with some fixed straight line, and $\theta + \delta\theta$ the angle which QO makes with the same line. The element PQ is acted on by a tension at P along the tangent at P, a tension at Q along the tangent at Q, and the resistance of the smooth curve which will be ultimately along PO.

Let s be the length of the curve measured from some fixed point up to P, and $PQ = \delta s$; let $R\delta s$ denote the resistance of the curve on PQ, T the tension at P, $T + \delta T$ the tension at Q. Suppose the element PQ to become rigid, and resolve the forces acting on it along the tangent and normal at P; then

$$T - (T + \delta T)\cos\delta\theta = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$R\delta s - (T + \delta T)\sin\delta\theta = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Now $$\cos\delta\theta = 1 - \frac{(\delta\theta)^2}{1.2} + \frac{(\delta\theta)^4}{\lfloor 4} - \ldots$$

hence (1) gives by division by $\delta\theta$

$$\frac{\delta T}{\delta\theta} - (T + \delta T)\left\{\frac{\delta\theta}{2} - \frac{(\delta\theta)^3}{\lfloor 4} + \ldots.\right\} = 0;$$

therefore ultimately

$$\frac{dT}{d\theta} = 0,$$

therefore $$T = \text{constant} \quad \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Also $\delta s = \rho \delta\theta$ ultimately, ρ being the radius of curvature at P, therefore, from (2), we have

$$R = \frac{T}{\rho} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

Since T is constant, the string will not be in equilibrium unless the forces pulling at its two ends are equal; this is usually assumed as self-evident in the theory of the pully.

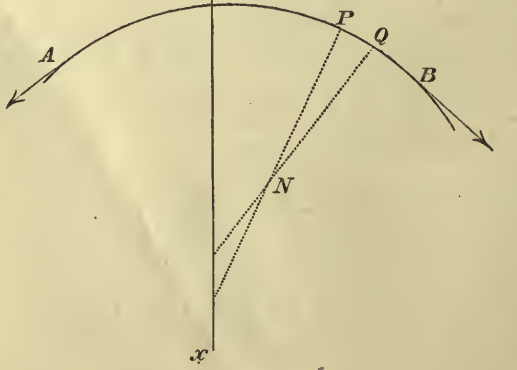
The *whole* pressure on the curve will be $\int R ds$; therefore by (4), the whole pressure

$$= \int \frac{T}{\rho} ds = \int T d\theta.$$

Since T is constant, $\int T d\theta = T\theta + \text{constant}$;

therefore the whole pressure $= T(\theta_2 - \theta_1)$, supposing θ_1 the value of θ at A, and θ_2 the value of θ at B.

Next suppose the weight of the string taken into account, and the plane of the string vertical.



Take the axis of y horizontal and that of x vertically downwards. The element PQ is acted on by a tension at P along the tangent at P, a tension at Q along the tangent at Q, the weight of the element vertically downwards, and the resistance of the smooth curve which will be ultimately along the normal at P. Let θ be the acute angle which the normal PN makes with the axis of x, $\theta + \delta\theta$ the angle which the normal QN makes with the axis of x. Let s be the length of the curve measured from some fixed point up to P, and $PQ = \delta s$; let T be the tension at P, and $T + \delta T$ the tension at Q; let $mg\delta s$ be the weight of the element, and $R\delta s$ the resistance of the smooth curve on the element. Suppose the element PQ to become rigid, and resolve the forces acting on it along the tangent and normal at P; then

$$T - (T + \delta T)\cos\delta\theta - mg\delta s \sin\theta = 0 \quad \ldots\ldots\ldots(5),$$

$$R\delta s - (T + \delta T)\sin\delta\theta - mg\delta s\cos\theta = 0 \quad \ldots\ldots\ldots(6).$$

From (5) we obtain ultimately

$$\frac{dT}{ds} = -mg\sin\theta \ldots\ldots\ldots\ldots\ldots\ldots(7),$$

and from (6)

$$R = \frac{T}{\rho} + mg\cos\theta \ldots\ldots\ldots\ldots\ldots\ldots(8),$$

where ρ is the radius of curvature of the curve at P.

Since the curve is supposed to be a known curve, s and ρ may be supposed known functions of θ; thus (7) and (8) will enable us to find T and R in terms of θ. Or we may express T and R in terms of the rectangular co-ordinates of the point P; for if we denote these co-ordinates by x and y, we have

$$\sin\theta = \frac{dx}{ds}, \cos\theta = \frac{dy}{ds};$$

thus (7) may be written

$$\frac{dT}{ds} = -mg\frac{dx}{ds};$$

therefore, if m be constant,

$$T = -mgx + C,$$

where C is some constant; the value of this constant will be known if the tension of the string be known at some given point, for example at A or at B.

Also from (8)

$$R = \frac{C - mgx}{\rho} + mg\frac{dy}{ds};$$

and ρ and $\frac{dy}{ds}$ will be known in terms of x and y since the curve is known.

In the above investigations we stated that the resistance of the curve on the element PQ acts *ultimately* along the normal at P; and in forming the equations of equilibrium of the element of the string we supposed the resistance to act strictly along the normal at P. It is easy to shew that no error is thus introduced. For at P the resistance is along the normal at P, and at Q the resistance is along the normal at Q, hence the resistance on the element PQ may be taken to be a force which acts in some direction intermediate between the directions of these two normals; suppose ψ the angle which its direction makes with that of the normal at P. We should then write $R\delta s \cos\psi$ instead of $R\delta s$ in the equations (2) and (6), where ψ is an angle less than $\delta\theta$; hence in the limit $\cos\psi = 1$ and equations (4) and (8) remain unchanged. Also the term $R\delta s \sin\psi$ must be introduced into equations (1) and (5); thus equation (1) becomes

$$T - (T + \delta T)\cos\delta\theta - R\delta s \sin\psi = 0;$$

therefore $$\frac{\delta T}{\delta\theta} - (T + \delta T)\left\{\frac{\delta\theta}{2} - \frac{(\delta\theta)^3}{\lfloor 4} + \ldots\right\} + R\frac{\delta s}{\delta\theta}\sin\psi = 0;$$

and ultimately $\frac{\delta s}{\delta\theta} = \rho$ and $\sin\psi = 0$; hence as before

$$\frac{dT}{d\theta} = 0.$$

Similarly we may shew that equation (7) remains true after the introduction of the term $R\delta s \sin\psi$ into equation (5).

195. *A string is stretched over a rough plane curve; to find the tension at any point and the pressure on the curve in the limiting position of equilibrium.*

First suppose the weight of the string neglected. See the first figure of Article 194.

The element PQ is acted on by a tension at P along the tangent at P, a tension at Q along the tangent at Q, the resistance of the curve which will be ultimately along the normal at P, and the friction which will be ultimately along the tangent at P and in the direction opposite to that in which the element is about to move. Let T denote the tension at P, $T+\delta T$ that at Q, $R\delta s$ the resistance, $\mu R\delta s$ the friction; suppose the string about to move from A towards B. Suppose the element PQ to become rigid, and resolve the forces acting on it along the tangent and normal at P; then

$$T + \mu R\delta s - (T + \delta T)\cos\delta\theta = 0 \quad \ldots\ldots(1),$$

$$R\delta s - (T + \delta T)\sin\delta\theta = 0 \quad \ldots\ldots(2).$$

From (1) we have ultimately

$$\frac{dT}{ds} = \mu R \quad \ldots\ldots(3),$$

and from (2),

$$\frac{T}{\rho} = R \quad \ldots\ldots(4),$$

where ρ is the radius of curvature at P;

therefore

$$\frac{\rho}{T}\frac{dT}{ds} = \mu, \text{ or } \frac{1}{T}\frac{dT}{d\theta} = \mu,$$

therefore

$$\log T = \mu\theta + \text{constant},$$

therefore

$$T = Ce^{\mu\theta}.$$

Let T_1 be the force which acts on the string at the end A, and therefore the value of T at this point; and let T_2 be the force at B; let θ_1 and θ_2 be the corresponding values of θ;

then

$$T_1 = Ce^{\mu\theta_1}, \quad T_2 = Ce^{\mu\theta_2};$$

therefore

$$\frac{T_2}{T_1} = e^{\mu(\theta_2 - \theta_1)},$$

and $$T = T_1 e^{\mu(\theta-\theta_1)} = T_2 e^{\mu(\theta-\theta_2)}.$$

Also $$\int R ds = \int \frac{T}{\rho}\, ds = \int T d\theta = T_1 \int e^{\mu(\theta-\theta_1)}\, d\theta$$

$$= \frac{T_1}{\mu} e^{\mu(\theta-\theta_1)} + \text{constant};$$

therefore the whole pressure on the curve

$$= \frac{T_1 e^{-\mu\theta_1}}{\mu} (e^{\mu\theta_2} - e^{\mu\theta_1}) = \frac{T_2 - T_1}{\mu}.$$

Next suppose the weight of the string taken into account, and the plane of the string vertical. Proceeding as in the second case of Art. 194, and supposing the string about to move from A to B, we have

$$T - (T + \delta T)\cos\delta\theta - mg\delta s \sin\theta + \mu R \delta s = 0 \ldots\ldots (5),$$

$$R\delta s - (T + \delta T)\sin\delta\theta - mg\delta s\cos\theta = 0 \ldots\ldots (6).$$

From (5) we obtain ultimately

$$\frac{dT}{ds} = \mu R - mg\sin\theta,$$

and from (6) $$R = \frac{T}{\rho} + mg\cos\theta;$$

therefore $$\frac{dT}{ds} = \frac{\mu T}{\rho} + mg(\mu\cos\theta - \sin\theta),$$

therefore $$\rho\frac{dT}{ds} - \mu T = mg(\mu\cos\theta - \sin\theta)\rho,$$

that is $$\frac{dT}{d\theta} - \mu T = mg(\mu\cos\theta - \sin\theta)\rho.$$

Thus we have a *differential equation* for finding T, and we may proceed in the ordinary way to obtain the solution. Multiply both sides of the last equation by $e^{-\mu\theta}$; thus

$$\frac{d}{d\theta}(Te^{-\mu\theta}) = mge^{-\mu\theta}(\mu\cos\theta - \sin\theta)\rho;$$

therefore $$Te^{-\mu\theta} = \int mge^{-\mu\theta}(\mu\cos\theta - \sin\theta)\rho d\theta.$$

Hence when ρ is known in terms of θ we shall only have to integrate a known function of θ in order to obtain the value of T in terms of θ.

196. *To form the equations of equilibrium of a string stretched over a smooth surface and acted on by any forces.*

Let s be the length of the string measured from some fixed point B to the point P; x, y, z the co-ordinates of P; δs the length of the element of the string between P and an adjacent point Q; $m\delta s$ the mass of the element; $R\delta s$ the resistance of the surface on this element, the direction of which will be ultimately the normal to the surface at P; let α, β, γ be the angles which the normal at P makes with the axes; $Xm\delta s$, $Ym\delta s$, $Zm\delta s$ the forces parallel to the axes acting on the element, exclusive of the resistance $R\delta s$. Hence, in the equations of Art. 187, for Xm we must put $Xm + R\cos\alpha$, and make similar substitutions for Ym and Zm; therefore

$$\frac{d}{ds}\left(T\frac{dx}{ds}\right) + Xm + R\cos\alpha = 0 \text{ ..... (1)},$$

$$\frac{d}{ds}\left(T\frac{dy}{ds}\right) + Ym + R\cos\beta = 0 \text{ ..... (2)},$$

$$\frac{d}{ds}\left(T\frac{dz}{ds}\right) + Zm + R\cos\gamma = 0 \text{ ..... (3)}.$$

Multiply (1) by $\frac{dx}{ds}$, (2) by $\frac{dy}{ds}$, and (3) by $\frac{dz}{ds}$, and add; then, since

$$\frac{dx}{ds}\cos\alpha + \frac{dy}{ds}\cos\beta + \frac{dz}{ds}\cos\gamma = 0,$$

because a tangent to the surface at any point is perpendicular to the normal at that point, we have, as in Art. 188,

$$\frac{dT}{ds} + m\left(X\frac{dx}{ds} + Y\frac{dy}{ds} + Z\frac{dz}{ds}\right) = 0 \text{ ..... (4)}.$$

Again, multiply (1) by $\cos\alpha$, (2) by $\cos\beta$, and (3) by $\cos\gamma$, and add; then

$$T\left\{\frac{d^2x}{ds^2}\cos\alpha + \frac{d^2y}{ds^2}\cos\beta + \frac{d^2z}{ds^2}\cos\gamma\right\}$$
$$+ m\{X\cos\alpha + Y\cos\beta + Z\cos\gamma\} + R = 0 \text{...(5)}.$$

Let $Fm\delta s$ be the resultant of $Xm\delta s$, $Ym\delta s$, $Zm\delta s$, and ψ the angle its direction makes with the normal to the surface at the point (x, y, z); then

$$X\cos\alpha + Y\cos\beta + Z\cos\gamma = F\cos\psi.$$

Let ρ be the radius of absolute curvature of the curve formed by the string at the point (x, y, z); α', β', γ' the angles its direction makes with the axes; ϕ the angle its direction makes with the normal to the surface; then

$$\frac{d^2x}{ds^2} = \frac{\cos\alpha'}{\rho}, \quad \frac{d^2y}{ds^2} = \frac{\cos\beta'}{\rho}, \quad \frac{d^2z}{ds^2} = \frac{\cos\gamma'}{\rho}.$$

Hence (5) becomes

$$\frac{T}{\rho}\cos\phi + Fm\cos\psi + R = 0 \ldots\ldots\ldots\ldots\ldots\ldots (6).$$

Let $u = 0$ be the equation to the surface; then

$$\frac{\cos\alpha}{\dfrac{du}{dx}} = \frac{\cos\beta}{\dfrac{du}{dy}} = \frac{\cos\gamma}{\dfrac{du}{dz}} = N \text{ suppose.}$$

Hence (1) may be written

$$T\frac{d^2x}{ds^2} + Xm + \frac{dT}{ds}\frac{dx}{ds} + RN\frac{du}{dx} = 0,$$

and (2) and (3) may be similarly expressed.

Eliminate $\dfrac{dT}{ds}$ and RN, and we obtain

$$\left(T\frac{d^2x}{ds^2} + Xm\right)\left(\frac{dy}{ds}\frac{du}{dz} - \frac{dz}{ds}\frac{du}{dy}\right)$$
$$+ \left(T\frac{d^2y}{ds^2} + Ym\right)\left(\frac{dz}{ds}\frac{du}{dx} - \frac{dx}{ds}\frac{du}{dz}\right)$$
$$+ \left(T\frac{d^2z}{ds^2} + Zm\right)\left(\frac{dx}{ds}\frac{du}{dy} - \frac{dy}{ds}\frac{du}{dx}\right) = 0.$$

If we put for T its value from (4), the resulting equation,

together with $u = 0$, will determine the curve formed by the string.

It appears from Art. 189 that the resultant of $Fm\delta s$ and $R\delta s$ must lie in the osculating plane of the curve at the point (x, y, z). If the direction of $Fm\delta s$ be always normal to the surface $u = 0$, then, since that of $R\delta s$ is also normal to the surface, it follows that the *normal to the surface lies in the osculating plane to the curve.* This we know to be a property of the lines of maximum or minimum length that can be drawn on a surface between two given points. Hence, when a string is stretched over a smooth surface and acted on only by forces which are in the directions of normals to the surface at their points of application, it forms the line of maximum or minimum length that can be drawn on the surface between the extreme points of its contact with the surface.

When $Fm\delta s$ is always normal to the surface, it follows from (4) that T is constant.

197. We will now give some miscellaneous theorems connected with the subject of flexible strings.

I. Required the abscissa of the centre of gravity of an assigned portion of any string at rest, supposing its ends fixed, and gravity the only force.

This may be obtained by the ordinary process of integration, or more simply in the following manner. Imagine any portion of the string to become rigid: then it is kept in equilibrium by its weight and the tensions at the ends; these tensions act in the directions of the tangents at the ends. Hence the centre of gravity of any portion must be vertically over the point of intersection of the tangents at the extremities of the portion.

II. Suppose in Art. 192 that the string is uniform, and that the force is attractive, and varies as the n^{th} power of the distance. Thus we may put $F = -\mu r^n$; therefore

$$\frac{C}{p} = \frac{m\mu r^{n+1}}{n+1} + \text{constant}.$$

In the particular case in which the constant here introduced is zero we can easily complete the solution of the problem.

We have

$$\frac{1}{p} = \frac{r^{n+1}}{\lambda},$$

where λ is put for $\frac{C(n+1)}{m\mu}$.

Put $\frac{1}{u}$ for r; then

$$u^2 + \left(\frac{du}{d\theta}\right)^2 = \frac{1}{\lambda^2 u^{2n+2}},$$

therefore

$$\left(\frac{du}{d\theta}\right)^2 = \frac{1 - \lambda^2 u^{2n+4}}{\lambda^2 u^{2n+2}},$$

therefore

$$\frac{d\theta}{du} = \frac{\lambda u^{n+1}}{\sqrt{(1 - \lambda^2 u^{2n+4})}}.$$

Therefore, by integration,

$$(n+2)\,\theta + \text{const.} = \sin^{-1}(\lambda u^{n+2});$$

therefore

$$\frac{\lambda}{r^{n+2}} = \sin\{(n+2)\,\theta + \text{constant}\}.$$

If we fix the position of the initial line so that r may have its least value where $\theta = 0$, we shall determine the constant, and obtain

$$r^{n+2} = \lambda \sec(n+2)\,\theta;$$

or denoting by a the value of r when $\theta = 0$,

$$r^{n+2} = a^{n+2} \sec(n+2)\,\theta.$$

III. Suppose a flexible string to be in equilibrium under the action of a central force. Imagine any portion of the string to become rigid: then it is kept in equilibrium by the tensions at the ends and the resultant of the action of the central force on the elements of the string; this resultant will be some single force acting through the centre of force. Thus the portion of the string may be considered to be in equilibrium under the action of three forces; and these forces will therefore meet at a point. Hence we obtain the following theorem: *The resultant action of the central force on any*

portion of the string is directed along the straight line which joins the centre of force with the point of intersection of the tangents at the ends of the portion.

IV. Suppose that a flexible string is in equilibrium under the action of a central force which varies as the distance. Let r be the distance of any point from the centre of force; x, y the co-ordinates of the point referred to axes having the centre of force as origin. Let the force on an element of the string of which the length is δs and mass $m\delta s$ situated at the point (x, y) be $\mu rm\delta s$; then this force can be resolved into $\mu xm\delta s$ and $\mu ym\delta s$ parallel to the axes of x and y respectively.

Hence the components, parallel to the axes of x and y, of the action of the central force on any portion of the string are $\mu\int xmds$ and $\mu\int ymds$ respectively, the integrations extending over the portion considered. Now if $\bar{x}$ and $\bar{y}$ be the co-ordinates of the centre of gravity of the portion, we have

$$\bar{x} = \frac{\int xmds}{\int mds}, \quad \bar{y} = \frac{\int ymds}{\int mds}.$$

Hence we obtain the following theorem: *The straight line which joins the centre of gravity of any portion of the string to the centre of force coincides with the direction of the resultant central force on the portion.*

Hence combining this theorem with that obtained in III. we obtain the following property of the flexible string which is in equilibrium under the action of a central force varying as the distance: *The centre of gravity of any portion lies on the straight line which joins the centre of force with the point of intersection of the tangents at the extremities of the portion.*

Thus by II. we see that the property here enunciated will hold for a uniform string in the form of the curve

$$r^3 = a^3 \sec 3\theta.$$

V. Two weights are connected by a string which passes over a rough horizontal cylinder in a plane perpendicular to the axis: it is required to determine the resultant of the normal actions between the string and the cylinder in the state bordering on motion.

The normal action on any element δs of the string may be denoted by $R\delta s$, and the friction on the element by $\mu R\delta s$; thus the friction on the element *bears a constant ratio to the normal action, and the directions of the two forces are at right angles.*

Let P be the resultant normal action, and suppose its direction to make an angle θ with the vertical; then the resultant friction will be μP, and its direction will make an angle $\frac{\pi}{2} - \theta$ with the vertical on the other side of it. Hence, supposing the string to become rigid, and resolving horizontally,

$$P \sin\theta - \mu P \sin\left(\frac{\pi}{2} - \theta\right) = 0.$$

Again, resolving vertically, and denoting by W the sum of the weights of the system which hangs over the cylinder, we have

$$P \cos\theta + \mu P \cos\left(\frac{\pi}{2} - \theta\right) - W = 0.$$

Hence we obtain $\tan\theta = \mu$,

$$P = W\cos\theta = \frac{W}{\sqrt{(1+\mu^2)}}.$$

VI. Suppose a heavy string which is not of uniform density and thickness to be suspended from two fixed points, and to be in equilibrium. Let t be the tension at any point, θ the angle which the tangent at that point makes with the horizon; then $t\cos\theta$ will be constant. For imagine any portion of the string to become rigid, then the only horizontal forces which act on it are the resolved parts of the tensions at each end; and these must therefore be equal in magnitude: therefore

$$t\cos\theta = \text{constant} = \tau \text{ suppose} \ldots\ldots\ldots\ldots (1).$$

Let w be the weight of the portion of the string contained between any fixed point and the variable point considered. Then by resolving the forces vertically we obtain in a similar manner

$$t\sin\theta - w = \text{constant};$$

therefore $$w = \tau\tan\theta + \text{constant} \ldots\ldots\ldots\ldots\ldots (2).$$

Again, proceeding as in Art. 193, that is resolving the forces which act on an element along the normal, we find

$$\frac{t}{\rho} - gm\cos\theta = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (3),$$

where $gm\delta s$ is taken as the weight of the element δs. Hence, from (1) and (3),

$$\frac{\tau}{\rho} = gm\cos^2\theta.$$

EXAMPLES.

1. In the common catenary shew that the weight of the string between the lowest point and any other point is the geometrical mean between the sum and difference of the tensions at the two points.

2. If α and β are the inclinations to the horizon of the tangents at the extremities of a portion of a common catenary, and l the length of the portion, shew that the height of one extremity above the other is

$$l\,\frac{\sin\dfrac{\alpha+\beta}{2}}{\cos\dfrac{\alpha-\beta}{2}};$$

the portion is supposed to be all on the same side of the lowest point.

3. A uniform heavy chain 110 feet long is suspended from two points in the same horizontal plane 108 feet asunder: shew that the tension at the lowest point is 1·477 times the weight of the chain nearly.

4. A uniform chain of length $2l$ is suspended from two fixed points in the same horizontal plane; $2a$ is the distance between the fixed points and c the length of chain whose weight is equal to the tension at the lowest point: shew that

when l is such that the tension at the points of suspension is the least possible that tension is equal to the weight of a length $\frac{al}{c}$ of the chain, and l and c are determined by

$$l = \tfrac{1}{2}c\left(e^{\frac{a}{c}} - e^{-\frac{a}{c}}\right), \qquad (l^2 + c^2)\, c^2 = a^2 l^2.$$

5. If a uniform chain be fixed at two points, and any number of links A, B, C, ... be at liberty to move along smooth horizontal lines in the same vertical plane, prove that the loops AB, BC, CD, ... will form themselves into curves which will all be arcs of the same catenary.

6. Three links of a chain A, B, and C are moveable freely along three rigid horizontal straight lines in the same vertical plane. If when A and C are pulled as far apart as possible, their horizontal distances from B are equal, shew that this will always be the case when they are held in any other position.

7. A chain hangs in equilibrium over two smooth points which are in a horizontal straight line and at a given distance apart: find the least length of the chain that equilibrium may be possible.

Result. The least length is he, where h is the given distance.

8. Prove that the exertion necessary to hold a kite diminishes as the kite rises higher, the force of the wind being independent of the height, and the pressure of the wind on the string being neglected.

9. A uniform heavy string rests on an arc of a smooth curve whose plane is vertical, shew that the tension at any point is proportional to its vertical height above the lowest point of the string. If the string rests on a parabola whose axis is vertical, determine the vertical distance of its ends below the highest point so that the pressure at this point may be equal to twice the weight of a unit of length of the string.

Result. The vertical distance is equal to half the latus rectum of the parabola.

10. One end of a uniform heavy chain hangs freely over the edge of a smooth table, and the other end passing over a fixed pully reaches to the same distance below the table as the pully is above it. Supposing half the chain to be on the table in the position of equilibrium, compare its whole length with the height of the pully.

Result. The length is to the height as $6 + 2\sqrt{3}$ is to 1.

11. A uniform heavy chain is fastened at its extremities to two rings of equal weights which slide on smooth rods intersecting in a vertical plane and inclined at the same angle α to the vertical: find the condition that the tension at the lowest point may be equal to half the weight of the chain; and in that case shew that the vertical distance of the rings from the point of intersection of the rods is

$$\frac{l}{2}\cot\alpha\log(1+\sqrt{2}),$$

where l is the length of the chain.

12. The density at any point of a catenary of variable density varies as the radius of curvature: determine the equation to the catenary.

Result. The curve in Art. 190.

13. A heavy cord with one end fixed to a point in the surface of a smooth horizontal cylinder is passed below the cylinder and carried round over the top, the other end being allowed to hang freely. Shew that unless the portion which hangs vertically be longer than the diameter of the cylinder, the cord will slip off, so as to hang down from the fixed point without passing below the cylinder.

14. If a uniform string hang in the form of a parabola by the action of normal forces only, the force at any point P varies as $(SP)^{-\frac{4}{3}}$, S being the focus.

15. If a string without weight touch a given cylinder in $\frac{1}{4}$th part of its circumference and in a plane perpendicular to its axis, what tension at one extremity will support a weight of 100 lbs. suspended at the other, friction being supposed to be $\frac{1}{10}$th part of the pressure? To what will this tension be reduced if the string is wound round $1\frac{1}{4}$th circumferences?

16. If $\mu = \frac{1}{4}$, and a string without weight passes twice round a post, prove, by taking approximate values of e and π, that any force will support another more than twenty times as great.

17. If two scales, one containing a weight P and the other a weight Q, be suspended by a string without weight over a rough sphere, and if Q be on the point of descending, then the weight $\frac{Q^2 - P^2}{P}$ put into the opposite scale will make that scale be on the point of descending.

18. Two equal weights P, P' are connected by a string without weight which passes over a rough fixed horizontal cylinder: compare the forces required to raise P according as P is pushed up or P' pulled down.

19. A, B, C are three rough pegs in a vertical plane: P, Q, R are the greatest weights which can be severally supported by a weight W, when connected with it by strings without weight passing over A, B, C, over A, B, and over B, C respectively: shew that the coefficient of friction at B is $\frac{1}{\pi} \log \frac{Q \,.\, R}{P \,.\, W}$.

20. A light thread, whose length is $7a$, has its extremities fastened to those of a uniform heavy rod whose length is $5a$, and when the thread is passed over a thin round peg, it is found that the rod will hang at rest, provided the point of support be anywhere within a space a in the middle of the thread: determine the coefficient of friction between the thread and the peg when the rod hangs in a position bordering on motion, and find its inclination to the horizon and the tensions of the different parts of the string.

Results. The coefficient of friction is determined by the equation $e^{\frac{1}{2}\mu\pi} = \frac{4}{3}$. The inclination of the rod to the horizon is $\cos^{-1} \frac{24}{25}$.

21. From a fixed point a given heavy uniform chain hangs down so that part of the chain rests on a rough horizontal plane: find the least length of chain that may be in contact with the plane.

22. A heavy chain of weight W rests entirely in contact with the arc of a rough closed vertical curve in a state bordering on motion. If $\tan\alpha$ be the coefficient of friction, shew that the resultant normal pressure on the curve is equal to $W\cos\alpha$, and that its direction makes an angle α with the vertical.

23. A heavy chain of length l rests partly on a rough horizontal table, and the remainder passing over the smooth edge of the table, (which is rounded off into the form of a semi-cylinder of radius a) hangs freely down: shew that if z be the least length on the table consistent with equilibrium,

$$z(\mu+1)=l-\tfrac{1}{2}\pi a+a.$$

24. A heavy uniform chain is hung round the circumference of a rough vertical circle of given radius. How much lower must one end of the chain hang than the other when it is on the point of motion?

Result. Let a be the length of the longer piece which hangs down, b the length of the shorter piece, r the radius of the circle, $\tan\beta$ the coefficient of friction; then

$$e^{\pi\tan\beta}=\frac{a-r\sin 2\beta}{b+r\sin 2\beta}.$$

25. A uniform beam of weight W is moveable about a hinge at one extremity, and has the other attached to a string without weight which, passing over a very small rough peg placed vertically above the hinge, and at a distance from it equal to the length of the beam, supports a weight P: shew that if θ be the inclination of the beam to the vertical when it is just on the point of falling, then

$$W\sin\tfrac{1}{2}\theta=Pe^{\frac{\mu(\pi+\theta)}{2}}.$$

Find also the pressure on the hinge.

26. One end of a heavy chain is attached to a fixed point A, and the other end to a weight which is placed on a rough horizontal plane passing through A, and the chain hangs through a slit in the horizontal plane. Shew that if l be the length of the chain, a the greatest distance of the weight

from A at which equilibrium is possible, μ the coefficient of friction, and n twice the ratio of the given weight to the weight of the chain,

$$\mu(1+n)\, e^{\mu\frac{a}{(1+n)}} = 1 + \sqrt{\{1+\mu^2(1+n)^2\}}.$$

27. A uniform string acted on by a central force assumes the form of an arc of a circle; determine the law of the force, the centre of force being on the circumference of the circle.

Result. The force varies inversely as the cube of the distance.

28. A smooth sphere rests upon a string without weight fastened at its extremities to two fixed points; shew that if the arc of contact of the string and sphere be not less than $2\tan^{-1}\frac{48}{55}$, the sphere may be divided into two equal portions by means of a vertical plane without disturbing the equilibrium.

29. Shew that if a chain exactly surrounds a smooth vertical circle, so as to be in contact at the lowest point without pressing, the whole pressure on the circle is double the weight of the chain, and the tension at the highest point is three times that at the lowest.

30. A smooth sphere of radius r and weight W is supported by resting on two equal strings without weight, which have their ends fixed at the same two points, both lying on the production of a horizontal diameter of the sphere. If the plane in which each string lies makes an angle α with the horizon, prove that the tension of each is $\frac{Wa}{8r}\operatorname{cosec}\alpha$; a being the distance between the points.

31. A uniform heavy chain hangs over two smooth pegs at a distance $2a$ apart in the same horizontal plane. When there is equilibrium, $2s$ is the length of the chain between the pegs, which hangs in the form of a catenary, c is the length of a portion of the chain whose weight is equal to the tension at the lowest point, and h the length of the end that hangs down vertically. If δs and δh be the small increments of s and h corresponding to a small uniform expansion of the chain, shew that $\delta s : \delta h = s.c - h.a : h.c - s.a$.

32. A uniform heavy chain is placed on a rough inclined plane; what length of chain must hang over the top of the plane, in order that the chain may be on the point of slipping up the plane?

33. A uniform rod of length b has its ends attached to the ends of a flexible string without weight of length a; this string is passed over a very small cylindrical peg, and when the rod hangs in its limiting position of equilibrium, the parts of the string on opposite sides of the peg are inclined to each other at an angle α. Shew that the coefficient of friction between the string and peg is

$$\frac{1}{\pi - \alpha} \log \frac{a + \sqrt{\{b^2 - (a^2 - b^2)\tan^2 \tfrac{1}{2}\alpha\}}}{a - \sqrt{\{b^2 - (a^2 - b^2)\tan^2 \tfrac{1}{2}\alpha\}}}.$$

34. AB, AC are two equal and uniform rods moveable about a fixed hinge at A, CB a uniform chain, equal in length to AB or AC and $\left(\frac{1}{n}\right)^{\text{th}}$ of its weight, connects the ends B and C; shew that in the position of equilibrium, the angle θ which either rod makes with the horizon is given approximately by the equation

$$\cos\theta = \tfrac{1}{2} - \frac{1}{4(n+1)^2},$$

n being large compared with unity.

35. A heavy uniform beam has its extremities attached to a string which passes round the arc of a rough vertical circle; if in the limiting position of equilibrium the beam be inclined at an angle of 60° to the vertical, and the portion of string in contact with the circle cover an arc of 270°, shew that the coefficient of friction is $\frac{1}{3\pi}\log 3$.

36. A uniform string just circumscribes a given smooth circle, and is attracted by a force varying as the distance to a point within the circle. Find the tension at any point, supposing it to vanish at the point nearest to the centre of force, and shew that the force at the greatest distance

$$= \frac{\text{whole pressure on circle}}{\text{mass of the string}}.$$

37. A heavy string whose length is $\frac{\pi}{2}a$ rests on the circumference of a rough vertical circle of radius a; if the string be in a position of limiting equilibrium, and if β be the angular distance of its highest extremity from the vertex of the circle, shew that

$$\tan\beta = \frac{(1-\mu^2)\,e^{\frac{\mu\pi}{2}} - 2\mu}{1-\mu^2+2\mu e^{\frac{\mu\pi}{2}}},$$

and explain this result when $(1-\mu^2)\,e^{\frac{\mu\pi}{2}} - 2\mu$ is negative.

Also if μ be such that $\beta = 0$, shew that the *whole pressure* on the curve is to the weight of the string as 2 is to $\pi\mu$.

38. Shew that if a uniform chain, laid upon a smooth table and fastened at the ends, is held in the form of a rectangular hyperbola by strings fastened one to each link and crossing each other at the centre of the hyperbola, the tensions of the strings will be equal.

CHAPTER XII.

FLEXIBLE EXTENSIBLE STRINGS.

198. In the preceding Chapter we considered the equilibrium of flexible *inextensible* strings; we now proceed to some propositions relative to flexible *extensible* strings. Such strings are also called *elastic* strings.

When a uniform extensible string is stretched by a force, it is found by experiment that, for small extensions, the extension varies as the product of the original length and the stretching force. Thus if T represent the force, l' the original length, l the stretched length,

$$l - l' = \frac{l'T}{\lambda},$$

where λ is some constant depending on the nature of the string.

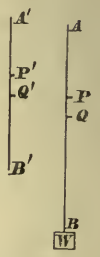
The fact expressed by this equation is called Hooke's law, from the name of its discoverer.

The quantity λ is sometimes called the *modulus of elasticity*.

In the equation $l - l' = \frac{l'T}{\lambda}$ if we put $T = \lambda$ we obtain $l = 2l'$; thus the modulus of elasticity for any uniform elastic string is equal to the tension which would be required to stretch that string to double its natural length, if the formula remained true for so large an extension.

199. *An elastic string has a weight attached to one end, it is fastened at the other and hangs vertically; determine the extension of the string, taking its own weight into account.*

Let $A'B'$ represent the natural length of the string; AB the stretched length. Let $A'P' = x'$, $P'Q' = \delta x'$. Suppose $A'P'$ stretched into AP, and $P'Q'$ into PQ; let $AP = x$, $PQ = \delta x$. Let $A'B' = a'$, w = the weight of the string, and W be the attached weight.



Let T be the tension at P, and $T + \delta T$ the tension at Q. Then the element PQ is acted on by the forces T and $T + \delta T$ at its ends, and by its own weight; its weight is the same as that of $P'Q'$, that is $\frac{\delta x'}{a'} w$;

therefore $$T + \delta T - T + \frac{\delta x'}{a'} w = 0,$$

or $$\frac{dT}{dx'} = -\frac{w}{a'} \text{ ultimately} \ldots\ldots\ldots\ldots\ldots\ldots (1);$$

therefore $$T = -\frac{wx'}{a'} + \text{constant}.$$

The value of the constant must be found by observing that when $x' = a'$, $T = W$; therefore

$$W = -w + \text{constant};$$

therefore $$T = W + w\left(1 - \frac{x'}{a'}\right) \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Also the element PQ may be considered ultimately uniform and stretched by a tension T; hence, by the experimental law,

$$\delta x = \delta x' \left(1 + \frac{T}{\lambda}\right) \ldots\ldots\ldots\ldots\ldots\ldots (3);$$

therefore $$\frac{dx}{dx'} = 1 + \frac{T}{\lambda},$$

$$= 1 + \frac{W}{\lambda} + \frac{w}{\lambda}\left(1 - \frac{x'}{a'}\right).$$

Integrate; thus

$$x = x'\left(1 + \frac{W + w}{\lambda}\right) - \frac{wx'^2}{2\lambda a'}.$$

No constant is required because $x = 0$ when $x' = 0$.

Let a denote the stretched length of the string; then putting $x' = a'$, we have

$$a = a'\left(1 + \frac{W+w}{\lambda}\right) - \frac{wa'}{2\lambda} = a'\left(1 + \frac{W+\frac{1}{2}w}{\lambda}\right).$$

Thus the extension is the same as would be produced if an elastic string of length a', the weight of which might be neglected, were stretched by a weight $W + \frac{1}{2}w$ at its end.

200. In the solution of the preceding problem we might have arrived at equation (2) by observing that the tension at any point must be equal to the weight of the string below that point together with W; but the method we adopted is more useful as a guide to the solution of similar problems. It is perhaps not superfluous to notice an error into which students often fall; since the element δx is acted on by a tension T at one end, and $T + \delta T$ or ultimately T at the other end, $2T$ is considered the stretching force, and instead of (3)

$$\delta x = \delta x'\left(1 + \frac{2T}{\lambda}\right)$$

is used. This would be of no consequence if *uniformly* adopted, for it would only amount to using $\frac{1}{2}\lambda$ instead of λ in (3); but mistakes arise from not adhering to one system or the other. It should be observed that if a string without weight be acted on by a force T at each end, it is in the same state of tension as if it were *fastened* at one end and acted on by a force T at the other.

201. The equations of Art. 187 and Art. 196 may be applied to an elastic string in equilibrium. They may also be modified as follows, if we wish to introduce the unstretched length of the string instead of the stretched length.

Let s' and $\delta s'$ represent the natural lengths which become s and δs by stretching; let $m'\delta s'$ be the mass of an element before stretching, and $m\delta s$ the mass of the same element after stretching; then

$$m\delta s = m'\delta s', \quad \delta s = \delta s'\left(1 + \frac{T}{\lambda}\right);$$

therefore

$$m\left(1 + \frac{T}{\lambda}\right) = m'.$$

Hence the first equation of equilibrium of Art. 187 may be written

$$\frac{d}{ds}\left(T\frac{dx}{ds}\right)+\frac{m'X}{1+\frac{T}{\lambda}}=0;$$

and the other two equations may be written similarly.

Equation (2) of Art. 188, or equation (4) of Art. 196 becomes

$$\left(1+\frac{T}{\lambda}\right)\frac{dT}{ds}+m'\left(X\frac{dx}{ds}+Y\frac{dy}{ds}+Z\frac{dz}{ds}\right)=0;\text{ therefore}$$

$$\lambda\left(1+\frac{T}{\lambda}\right)^2+2m'\int\left(X\frac{dx}{ds}+Y\frac{dy}{ds}+Z\frac{dz}{ds}\right)ds=\text{constant},$$

provided m' be constant; that is, provided the product of the density into the area of the section of the string in its unstretched state is constant: see Art. 187.

Since $\left(1+\frac{T}{\lambda}\right)^2=\left(\frac{ds}{ds'}\right)^2$, the last equation may be used to connect s and s', and thus find the extension of the string.

202. We may apply the preceding Article to the case in which the weight of the string is the only force acting on it, the string being supposed originally uniform, and fixed at two points.

In this case $X=0$, $Y=-g$, $Z=0$, as in Art. 190; therefore

$$\frac{d}{ds}\left(T\frac{dx}{ds}\right)=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$\left(1+\frac{T}{\lambda}\right)\frac{d}{ds}\left(T\frac{dy}{ds}\right)-mg'=0\ldots\ldots\ldots\ldots\ldots(2).$$

From (1) $T\frac{dx}{ds}=$ a constant $=m'cg$ suppose;

therefore $$T=m'cg\sec\psi\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

where ψ is the angle which the tangent to the curve at the point (x, y) makes with the axis of x.

Hence (2) gives

$$\left(1+\frac{m'cg}{\lambda}\sec\psi\right)\frac{d\tan\psi}{ds}=\frac{1}{c}\ldots\ldots\ldots\ldots(4),$$

therefore $$\left(1+\frac{m'cg}{\lambda}\sec\psi\right)\frac{d\tan\psi}{dx}\frac{dx}{ds}=\frac{1}{c},$$

thus $$\cos\psi\left(1+\frac{m'cg}{\lambda}\sec\psi\right)\frac{d\tan\psi}{dx}=\frac{1}{c},$$

that is $$\frac{1}{\cos\psi}\frac{d\psi}{dx}+\frac{m'cg}{\lambda}\frac{d\tan\psi}{dx}=\frac{1}{c};$$

therefore, by integration,

$$\int\frac{1}{\cos\psi}\frac{d\psi}{dx}dx+\frac{m'cg}{\lambda}\tan\psi=\frac{x}{c}$$

and $$\int\frac{1}{\cos\psi}\frac{d\psi}{dx}dx=\int\frac{d\psi}{\cos\psi}=\log\frac{1+\sin\psi}{\cos\psi};\ \text{thus}$$

$$\log\frac{1+\sin\psi}{\cos\psi}+\frac{m'cg}{\lambda}\tan\psi=\frac{x}{c}\ldots\ldots\ldots\ldots(5).$$

No constant is required in the integration if we suppose the axis of y to pass through the lowest point of the curve, for there $\psi=0$.

From (4) we may deduce

$$\sin\psi\left(1+\frac{m'cg}{\lambda}\sec\psi\right)\frac{d\tan\psi}{dy}=\frac{1}{c}\ldots\ldots\ldots\ldots(6);$$

therefore, by integration,

$$\sec\psi+\frac{m'cg}{2\lambda}\tan^2\psi=\frac{y}{c}\ldots\ldots\ldots\ldots\ldots\ldots(7).$$

No constant is required in the integration if we suppose the origin of co-ordinates to be at the distance c below the lowest point of the curve.

From equation (7) we may find $\sec\psi$ in terms of y, and then $\cos\psi$ and $\sin\psi$ can also be found; thus by substituting in (5) we could obtain the equation between x and y: this equation however would be very complex.

In a particular case we may easily obtain an approximate value of y in terms of x. Let $\lambda = m'gl$; then (5) may be written

$$\frac{1+\sin\psi}{\cos\psi} = e^{\frac{x}{c} - \frac{c}{l}\tan\psi},$$

therefore

$$\frac{\cos\psi}{1+\sin\psi} = e^{-\frac{x}{c} + \frac{c}{l}\tan\psi};$$

therefore by addition and reduction

$$\frac{2}{\cos\psi} = e^{\frac{x}{c} - \frac{c}{l}\tan\psi} + e^{-\frac{x}{c} + \frac{c}{l}\tan\psi},$$

therefore

$$\tan^2\psi = \tfrac{1}{4}\left(e^{\frac{x}{c} - \frac{c}{l}\tan\psi} - e^{-\frac{x}{c} + \frac{c}{l}\tan\psi}\right)^2.$$

Now suppose $\frac{c}{l}$ is a very small quantity, put u for $\frac{1}{2}\left(e^{\frac{x}{c}} - e^{-\frac{x}{c}}\right)$ and v for $\frac{1}{2}\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right)$; then the last equation gives

$$\tan\psi = u - \frac{cv}{l}\tan\psi + \frac{c^2u}{1.2l^2}\tan^2\psi - \frac{c^3v}{3l^3}\tan^3\psi + \ldots;$$

from this we can find $\tan\psi$ approximately, and then $\sec\psi$ will be known approximately, and by substituting in (7) we shall obtain approximately y in terms of x.

Equation (2) may also be written

$$\frac{d}{ds'}\left(T\frac{dy}{ds}\right) = m'g,$$

therefore

$$\frac{d}{ds'}\left(m'cg\frac{dy}{dx}\right) = m'g;$$

therefore, by integration,

$$\frac{dy}{dx} = \frac{s'}{c};$$

here s' denotes the natural length of that portion of the string which is between the lowest point and the point (x, y).

Hence for $\tan\psi$ in (5) and (7) we may put $\frac{s'}{c}$, and make corresponding substitutions for $\sin\psi$ and $\cos\psi$. Thus (7) becomes

$$\sqrt{(c^2+s'^2)}+\frac{m'gs'^2}{2\lambda}=y \quad \ldots\ldots\ldots\ldots\ldots\ldots (8).$$

As an example of these formulæ suppose that a heavy uniform elastic string hangs in equilibrium over two smooth pegs in a horizontal plane, and let it be required to find the depth of the ends of the string below the vertex of the curved portion.

From (3) the tension at any point of the curve is

$$m'g\sqrt{(c^2+s'^2)}.$$

Let l' be the natural length of the portion which hangs over one of the pegs; then the weight of this portion is $m'gl'$. Let s' denote the unstretched length of the portion between the vertex and one peg; then by equating the two expressions for the tension, we have

$$m'gl'=m'g\sqrt{(c^2+s'^2)},$$

therefore
$$l'=\sqrt{(c^2+s'^2)} \quad \ldots\ldots\ldots\ldots\ldots\ldots (9);$$

thus from (8) and (9)

$$l'+\frac{m'g\,(l'^2-c^2)}{2\lambda}=y \quad \ldots\ldots\ldots\ldots\ldots\ldots (10).$$

Suppose l to be the length to which a string of natural length l' hanging vertically would be stretched; then by Art. 199,

$$l=l'\left(1+\frac{m'gl'}{2\lambda}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots (11).$$

By (10) and (11)

$$l=y+\frac{m'gc^2}{2\lambda}.$$

Thus the end of the string descends to the depth $\frac{m'gc^2}{2\lambda}$ below the axis of x, and therefore to the depth $c\left(1+\frac{m'cg}{2\lambda}\right)$ below the vertex of the curve.

EXAMPLES.

1. Two equal heavy beams, AB, CD, are connected diagonally by similar and equal elastic strings AD, BC; and AB is fixed in a horizontal position: shew that if the natural length of each string equals AB, and the elasticity be such that the weight of AB would stretch the string to three times its natural length, then

$$\frac{1}{AB}=\frac{1}{BC}+\frac{1}{AC}.$$

2. An elastic string will just reach round two pegs in a horizontal plane; a ring whose weight would double the length of the string hanging from a point is slung on it: shew that if θ be the inclination of two portions of the string to the horizon,

$$\sin 2\theta = 2(\sqrt{2}-1).$$

3. An elastic string has its ends attached to those of a uniform beam of the same length as the unstretched string, the weight of the beam being such as would stretch the string to twice its natural length; shew that when the system is hung up by means of the string on a smooth peg, the inclination θ of the string to the vertical will be given by the equation

$$\tan\theta + 2\sin\theta - 2 = 0.$$

4. Three equal circular discs are placed in contact in a vertical plane with their centres in the same horizontal line, and an endless elastic cord wound alternately above and below them, so as to touch every point of their circumferences without being stretched beyond its natural length. When the support of the middle disc is removed, the centres of the

three form a right-angled triangle. Shew that the modulus of elasticity of the cord is $\frac{W}{2} \cdot \frac{3\pi}{4-\pi}$, W being the weight of the disc.

5. A fine elastic string is tied round two equal cylinders whose surfaces are in contact and axes parallel, the string not being stretched beyond its natural length; one of the cylinders is turned through two right angles, so that the axes are again parallel: find the tension of the string, supposing a weight of 1 lb. would stretch it to twice its natural length.

Result. $\frac{\pi-2}{\pi+2}$ of a lb.

6. Two equal and similar elastic strings AC, BC, fixed at two points A, B in the same horizontal line, support a given weight at C. The extensibility and original lengths of the strings being given, find an equation for determining the angle at which each string is inclined to the horizon, and deduce an approximate value of the angle when the extensibility is very small.

7. Six equal rods are fastened together by hinges at each end, and one of the rods being supported in a horizontal position the opposite one is fastened to it by an elastic string joining their middle points. Supposing the modulus of elasticity is equal to the weight of each rod, find the original length of the string in order that the hexagon may be equiangular in its position of equilibrium.

Result. $\frac{a\sqrt{3}}{4}$, where a is the length of a rod.

8. An unstretched elastic string without weight has n equal weights attached to it at equal distances, and is then suspended from one end. Prove that the increase of length is half what it would be if the same string were stretched by a weight equal to $n+1$ of the former hanging at one end.

9. Three equal cylindrical rods are placed symmetrically round a fourth of the same radius, and the bundle is then surrounded by two equal elastic bands at equal distances

from the two ends; if each band when unstretched would just pass round one rod, and a weight of 1 lb. would just stretch it to twice its natural length, shew that it would require a force of 9 lbs. to extract the middle rod, the coefficient of friction being equal to $\frac{1}{6}\pi$.

10. Two elastic strings are just long enough to fit on a sphere without stretching; they are placed in two planes at right angles to each other, and the sphere is suspended at their point of intersection. If 2θ be the angle subtended at the centre by the arc which is unwrapped, shew that

$$\theta^4 = \frac{3\pi}{4}\frac{W}{\lambda},$$

θ being supposed small.

11. In the common catenary, if the string be slightly extensible, shew that its whole extension will be proportional to the product of its length and the height of its centre of gravity above the directrix.

12. A uniform rough cylinder is supported with its axis horizontal by an elastic string without weight; the string lies in the plane which is perpendicular to the axis of the cylinder, and passes through its centre of gravity; the ends of the string are attached to points which are in the same horizontal plane above the cylinder and at a distance equal to the diameter of the cylinder. Find how much the string is stretched.

Result. Let $2W$ be the weight of the cylinder, a the radius of the cylinder, b' the natural length of each vertical portion of the string; then the extension is

$$\frac{2b'W}{\lambda} + \frac{2a}{\mu}\log\frac{\lambda + We^{\frac{\mu\pi}{2}}}{\lambda + W}.$$

13. A heavy string very slightly elastic is suspended from two points in the same horizontal plane; shew that if c, l be the lengths of unstretched string whose weights are respectively equal to the tension at the lowest point and the modulus of elasticity, the equation to the catenary will be very approximately

$$y = \frac{c}{2}\left\{e^{\frac{x}{c}} + e^{-\frac{x}{c}} - \frac{c}{4l}\left(e^{\frac{x}{c}} - e^{-\frac{x}{c}}\right)^2\right\}.$$

14. A weight P just supports another weight Q by means of a fine elastic string passing over a rough cylinder whose axis is horizontal. If λ be the modulus of elasticity, μ the coefficient of friction, and a the radius of the cylinder, shew that the extension of that part of the string which is in contact with the cylinder is

$$\frac{a}{\mu}\log\frac{Q+\lambda}{P+\lambda}.$$

15. A sphere placed on a horizontal plane is divided by a vertical plane into two equal parts, which are just held together by an elastic string, which passes round the greatest horizontal section: find the original length of the string.

Result. $\dfrac{32\lambda\pi a}{16\lambda+3W}$.

16. Four equal heavy rods are fastened to one another by hinges so as to form a square $ABCD$; A and C are connected by an elastic string whose natural length is equal to the diagonal AC, and the system is suspended from the point A: find the position of equilibrium.

Result. Let W be the weight of a rod, θ the inclination of each rod to the vertical; then

$$\cos\theta=\frac{1}{\sqrt{2}}\left(1+\frac{2W}{\lambda}\right).$$

17. An elastic band, whose unstretched length is $2a$, is placed round four rough pegs A, B, C, D, which constitute the angular points of a square whose side is a; if it be taken hold of at a point P, between A and B, and pulled in the direction AB, shew that it will begin to slip round A and B at the same time if

$$AP=\frac{a}{1+e^{\frac{\mu\pi}{2}}}.$$

18. An elastic string without weight of variable thickness is extended by a given force: find the whole extension.

19. An elastic string whose density varies as the distance from one end, is suspended by that end and stretched by its

own weight. If W be the weight of the string, l' its unstretched length, l its stretched length, shew that

$$l = l'\left(1 + \frac{2W}{2\lambda}\right).$$

20. A circular elastic string is placed on a smooth sphere so that the whole string is in one horizontal plane; the string subtends when unstretched an angle 2α at the centre, and an angle 2θ when in a position of equilibrium: shew that

$$\sin\theta = \sin\alpha\left(1 + \frac{a}{c}\sin\alpha\tan\theta\right),$$

where a = radius of sphere, and c depends on the nature of the string.

21. A heavy uniform elastic string rests horizontally on a portion of a surface of revolution, of which the axis is vertical, in every position: prove that the generating curve is a parabola a diameter of which is the axis of revolution.

22. A heavy elastic string surrounds a smooth horizontal cylinder, so that the surface of the cylinder is subject to no pressure at the lowest point; find the pressure at any point of the cylinder, and the tension of the string; its modulus of elasticity being equal to the weight of a portion of string the natural length of which is $\frac{3}{8}$ of the diameter of the cylinder.

23. A uniform heavy elastic string, whose natural length is a, is stretched and placed in equilibrium on a rough inclined plane; find the tension at any point, and shew that the direction of the friction changes at a point of the string, the natural distance of which from the upper end is

$$\frac{a}{2}\left(1 + \frac{\tan\alpha}{\mu}\right),$$

where α is the inclination of the plane to the horizon.

24. A heavy elastic cord is passed through a number of fixed smooth rings. Shew that in the position of equi-

librium its extremities will lie in the same horizontal plane. The same will also be the case if the cord rest upon any smooth surface.

25. An elastic string is laid on a cycloidal arc, the plane of which is vertical and vertex upwards, and when stretched by its own weight is in contact with the whole of the cycloid; the modulus of elasticity is the weight of a portion of the string whose natural length is twice the diameter of the generating circle: find the natural length of the string.

Result. It is equal to the circumference of the generating circle.

CHAPTER XIII.

ATTRACTION.

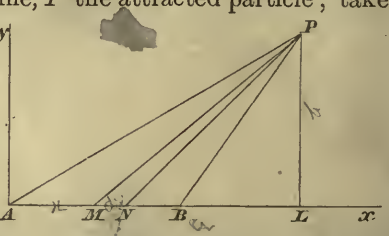
203. It appears from considerations which are detailed in works on Physical Astronomy, that two particles of matter placed at any sensible distance apart attract each other with a force directly proportional to the product of their masses, and inversely proportional to the square of their distance.

Suppose then a particle to be attracted by all the particles of a body; if we resolve the attraction of each particle of the body into components parallel to fixed rectangular axes, and take the sum of the components which act in a given direction, we obtain the resolved attraction of the whole body on the particle in that direction, and can thus ascertain the resultant attraction of the body in magnitude and direction. We shall give some particular examples, and then proceed to general formulæ.

204. *To find the attraction of a uniform straight line on an external point.*

By a straight line we understand a cylinder such that the section perpendicular to its axis is a curve every chord of which is indefinitely small.

Let AB be the straight line, P the attracted particle; take A for the origin, and AB for the direction of the axis of x. Draw PL perpendicular to Ax; let $AB = l$, $AL = a$, $PL = b$. Let M and N be adjacent points in the straight line, $AM = x$, $MN = \delta x$. If ρ be the density of the line, and κ the area of a section perpendicular



to its length, the mass of the element is $\rho\kappa\delta x$. Let m be the mass of P; then the attraction of the element MN on P is (Art. 203)

$$\frac{\mu m\, \rho\kappa\, \delta x}{(PM)^2},$$

where μ is some constant quantity. Hence, the resolved part of the attraction of the element parallel to the axis of x, is

$$\frac{\mu m\, \rho\kappa\, \delta x}{PM^2} \cdot \frac{ML}{PM} \text{ or } \frac{\mu m\, \rho\kappa\, (a-x)\, \delta x}{\{b^2+(a-x)^2\}^{\frac{3}{2}}}.$$

Also the resolved part of the attraction of the element parallel to the axis of y, is

$$\frac{\mu m\, \rho\kappa\, \delta x}{PM^2} \cdot \frac{PL}{PM} \text{ or } \frac{\mu m\, \rho\kappa b \delta x}{\{b^2+(a-x)^2\}^{\frac{3}{2}}}.$$

Let X and Y be the resolved parts of the attraction of the straight line, parallel to the axes of x and y respectively; then

$$X = \mu m\, \rho\kappa \int_0^l \frac{(a-x)\, dx}{\{b^2+(a-x)^2\}^{\frac{3}{2}}},$$

$$Y = \mu m\, \rho\kappa \int_0^l \frac{b dx}{\{b^2+(a-x)^2\}^{\frac{3}{2}}}.$$

Now
$$\int \frac{(a-x)\, dx}{\{b^2+(a-x)^2\}^{\frac{3}{2}}} = \frac{1}{\{b^2+(a-x)^2\}^{\frac{1}{2}}};$$

therefore
$$\int_0^l \frac{(a-x)\, dx}{\{b^2+(a-x)^2\}^{\frac{3}{2}}} = \frac{1}{\{b^2+(a-l)^2\}^{\frac{1}{2}}} - \frac{1}{\{b^2+a^2\}^{\frac{1}{2}}} \ldots(1);$$

$$\int \frac{dx}{\{b^2+(a-x)^2\}^{\frac{3}{2}}} = -\frac{a-x}{b^2\{b^2+(a-x)^2\}^{\frac{1}{2}}};$$

therefore

$$\int_0^l \frac{b dx}{\{b^2+(a-x)^2\}^{\frac{3}{2}}} = \frac{1}{b}\left[\frac{a}{(b^2+a^2)^{\frac{1}{2}}} - \frac{a-l}{\{b^2+(a-l)^2\}^{\frac{1}{2}}}\right] \ldots\ldots(2).$$

Hence, putting f for $\mu\rho\kappa$, we have

$$X = fm\left\{\frac{1}{PB} - \frac{1}{PA}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (3),$$

$$Y = \frac{fm}{PL}\left\{\frac{AL}{PA} - \frac{BL}{PB}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (4).$$

Let $APL = \alpha$, $BPL = \beta$, $APB = \gamma$; then

$$X = \frac{fm}{PL}(\cos\beta - \cos\alpha), \qquad Y = \frac{fm}{PL}(\sin\alpha - \sin\beta);$$

therefore $\sqrt{(X^2 + Y^2)} = \frac{fm}{PL}\sqrt{\{(\cos\beta - \cos\alpha)^2 + (\sin\alpha - \sin\beta)^2\}}$

$$= \frac{fm}{PL}\sqrt{(2 - 2\cos\gamma)} = \frac{2fm}{PL}\sin\tfrac{1}{2}\gamma \ldots (5).$$

This gives the magnitude of the resultant attraction. Also

$$\frac{X}{Y} = \frac{\cos\beta - \cos\alpha}{\sin\alpha - \sin\beta} = \tan\frac{\alpha + \beta}{2} \ldots\ldots\ldots (6).$$

This shews that the direction of the resultant attraction bisects the angle APB.

If L fall *between* A and B, it will be seen from (1) and (2) that the expression for X in (3) remains unchanged, but that for Y in (4) is changed to

$$\frac{fm}{PL}\left\{\frac{AL}{PA} + \frac{BL}{PB}\right\}.$$

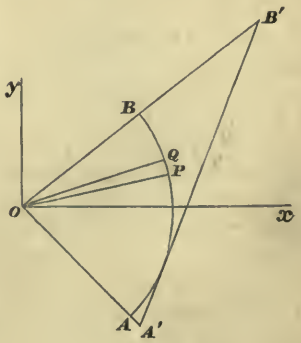
This will not affect the result in (5), and the direction of the resultant will still bisect the angle APB.

From the investigation it appears that X is the resolved attraction parallel to the axis of x in the direction of the negative axis of x, and Y the resolved attraction parallel to the axis of y also in the negative direction.

The attraction of the straight line may also be obtained by modifications of the process, which will furnish exercise for the student; for example, the origin of co-ordinates may be put at the attracted particle, and the axis of y taken parallel to the straight line.

205. In the above investigation we have taken m to denote the mass of the attracted particle; in future we shall always suppose the mass of the attracted particle to be denoted by unity. In order to form a precise idea of the quantity μ, we may suppose two particles each having its mass equal to the unit of mass, then μ will be the whole force which one of these exerts on the other when the distance between them is the unit of length. As, however, by properly choosing the unit of mass we may make $\mu = 1$, we shall not in future consider it necessary to introduce μ.

206. *To find the attraction of a circular arc on a particle situated at the centre of the circle.*



Let AB be any circular arc; through O the centre of the circle draw a line bisecting the angle AOB, and take this line for the axis of x. Let $POx = \theta$, $QOP = \delta\theta$, $AOB = 2\alpha$, $OB = r$. The attraction of the element PQ resolved parallel to the axes of x and y respectively is, if ρ and κ have the same meaning as in Art. 204,

$$\frac{\kappa\rho r\delta\theta}{r^2}\cos\theta \quad \text{and} \quad \frac{\kappa\rho r\delta\theta}{r^2}\sin\theta;$$

therefore

$$X = \frac{\kappa\rho}{r}\int_{-\alpha}^{+\alpha}\cos\theta d\theta = \frac{2\kappa\rho}{r}\sin\alpha,$$

$$Y = \frac{\kappa\rho}{r}\int_{-\alpha}^{+\alpha}\sin\theta d\theta = 0.$$

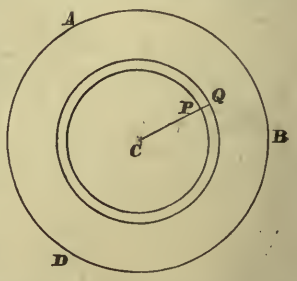
By comparing these results with those in Art. 204, it appears that the attraction of a circular arc on a particle at the centre is the same in magnitude and direction as that of any straight line $A'B'$ which *touches* the arc AB and is terminated by the lines OA and OB produced, the arc and the straight line being supposed to have the same density, and the areas of their transverse sections equal.

If OP and OQ be produced to meet the straight line $A'B'$ in points P' and Q' respectively, it may be shewn that the attraction of the element $P'Q'$ on a particle at O is equal to that of PQ, and in this manner we might prove what we have just shewn, that the attractions of AB and $A'B'$ on a particle at O are equal and coincident. This proposition is given in Earnshaw's *Dynamics*, p. 326.

It easily follows, that if a particle be attracted by the three sides of a triangle, it will be in equilibrium if it be placed at the centre of the circle inscribed in the triangle.

207. *To find the attraction of a uniform circular lamina on a particle situated in a straight line drawn through the centre of the lamina at right angles to its plane.*

Suppose C the centre of the circle DAB, the plane of the paper coinciding with one face of the lamina, and the attracted particle being in a straight line drawn through C perpendicular to the lamina and at a distance c from C. Describe from the centre C two adjacent concentric circles, one with radius $CP = r$, and the other with radius $CQ = r + \delta r$. Let κ denote the thickness of the lamina, which is supposed to be an indefinitely small quantity, then the mass of the circular ring contained between the adjacent circles is $2\pi\rho\kappa r\delta r$. Every particle in this circular ring is at a distance $\sqrt{(c^2 + r^2)}$ from the attracted particle; also the resultant attraction of the ring is in the straight line through C at right angles to the lamina, and is equal to



$$\frac{2\pi\rho\kappa r\delta r}{c^2 + r^2} \cdot \frac{c}{\sqrt{(c^2 + r^2)}},$$

the factor $\dfrac{c}{\sqrt{(c^2 + r^2)}}$ being the multiplier necessary in order to resolve the attraction of any element of the ring along the normal to the lamina through C.

Hence, the resultant attraction of the whole lamina is

$$2\pi\rho\kappa c\int_0^b \frac{rdr}{(c^2+r^2)^{\frac{3}{2}}},$$

where b is the radius of the boundary of the lamina.

Now $$\int \frac{rdr}{(c^2+r^2)^{\frac{3}{2}}} = -\frac{1}{\sqrt{(c^2+r^2)}};$$

therefore $$\int_0^b \frac{rdr}{(c^2+r^2)^{\frac{3}{2}}} = \frac{1}{c} - \frac{1}{\sqrt{(c^2+b^2)}};$$

therefore the resultant attraction

$$= 2\pi\rho\kappa c\left\{\frac{1}{c} - \frac{1}{\sqrt{(c^2+b^2)}}\right\}.$$

If we suppose b to become infinite, we obtain for the attraction of an infinite lamina on an external particle, the expression $2\pi\rho\kappa$, which is independent of the distance of the attracted particle from the lamina.

From the last result we can deduce the resultant attraction of a uniform plate of finite thickness, but of infinite extent, on an external particle. For, suppose the plate divided into an indefinitely large number of laminæ, each of the thickness κ; then the attraction of each lamina acts in a straight line through the attracted particle perpendicular to the surfaces of the plate, and is equal to $2\pi\rho\kappa$. Hence, the resultant attraction will be found by adding the attractions of the laminæ, and will be $2\pi\rho h$, if h be the thickness of the plate.

If a particle be placed *on* the exterior surface of an infinite plate, the result just found will express the attraction of the plate on the particle. If it be placed in the interior of the plate at a distance h from one of the bounding planes and h' from the other, the resultant attraction will be $2\pi\rho\,(h'-h)$ *towards* the latter plane.

208. By means of the preceding Article we can find the resultant attraction of a uniform right circular cylinder

on a particle situated on its axis. Suppose the cylinder divided into an indefinitely large number of laminæ by planes perpendicular to its axis; let x be the distance of a lamina from the attracted particle, δx the thickness of the lamina, b the radius of the cylinder; then the attraction of the lamina is

$$2\pi\rho\left\{1-\frac{x}{\sqrt{(x^2+b^2)}}\right\}\delta x.$$

Suppose the attracted particle *outside* the cylinder at a distance c from it; let h be the height of the cylinder: then the resultant attraction of the cylinder

$$=2\pi\rho\int_c^{c+h}\left\{1-\frac{x}{\sqrt{(x^2+b^2)}}\right\}dx$$

$$=2\pi\rho\left[h-\sqrt{\{(c+h)^2+b^2\}}+\sqrt{(c^2+b^2)}\right].$$

If we suppose $c=0$ so that the particle is on the surface of the cylinder the resultant attraction is

$$2\pi\rho\left\{h-\sqrt{(h^2+b^2)}+b\right\}.$$

209. To find the attraction of a uniform right circular *cone* on a particle at its vertex, we begin with the expression

$$2\pi\rho\left\{1-\frac{x}{\sqrt{(x^2+b^2)}}\right\}\delta x,$$

for the attraction of a lamina of the cone. Also, if α be the semivertical angle of the cone, we have

$$\frac{x}{\sqrt{(x^2+b^2)}}=\cos\alpha;$$

hence, the resultant attraction

$$=2\pi\rho\,(1-\cos\alpha)\int_0^h dx=2\pi\rho\,(1-\cos\alpha)\,h;$$

where h is the height of the cone.

It is easily seen that the same expression holds for the attraction of the frustum of a cone on a particle situated at the vertex of the complete cone, h representing in this case the height of the frustum.

If the cone be an *oblique* cone the base of which is any plane figure it is still true that the attraction of a frustum on a particle at the vertex varies as the thickness of the frustum. Consider two parallel laminæ of the same indefinitely small thickness at different distances from the vertex of such a cone, then the attractions of these laminæ on the particle at the vertex will be the same. For take any indefinitely small element of area on the surface of one of the laminæ, and let a conical surface be formed by straight lines which pass through the perimeter of this area and through the attracted particle; this conical surface will intercept elements in the two laminæ which are bounded by similar plane figures. Since the thicknesses of the laminæ are equal, the masses of the elements will vary as the squares of their distances from the attracted particle, and thus they will exert *equal attractions* on this particle. The same result holds for every corresponding pair of elements in the two laminæ, and thus the two laminæ exert on the particle at the vertex attractions which are equal in amount and coincident in direction. From this it follows that the attraction of a frustum varies as its thickness.

210. We have hitherto considered the attracting body to be of uniform density, but considerable variety may be introduced into the questions by various suppositions as to the law of density. Suppose, for instance, that in the case of the circular lamina in Art. 207 the density at any point of the lamina is $\phi(r)$, where r is the distance of that point from the centre; $\phi(r)$ must then be put instead of ρ in Art. 207 and must be placed under the integral sign. Therefore the attraction of the lamina will be

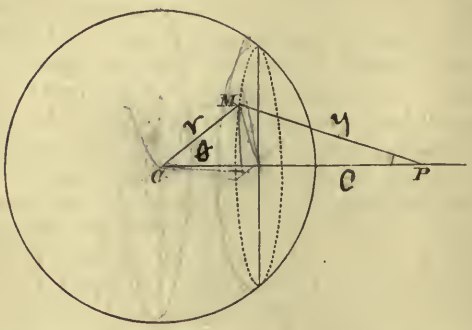
$$2\pi c\kappa \int_0^b \frac{\phi(r)\, r dr}{(c^2+r^2)^{\frac{3}{2}}}.$$

If $\phi(r) = \frac{\sigma}{r}$, where σ is a constant, the result is

$$2\pi c\kappa\sigma \int_0^b \frac{dr}{(c^2+r^2)^{\frac{3}{2}}}, \quad \text{or} \quad \frac{2\pi\kappa\sigma b}{c\,(c^2+b^2)^{\frac{1}{2}}}.$$

211. *To find the resultant attraction of an assemblage of particles constituting a homogeneous spherical shell of very small thickness on a particle outside the shell.*

Let C be the centre of the shell, M any particle of it, P the



attracted particle. Let $CM = r$, $PM = y$, $CP = c$, $\theta =$ the angle PCM, $\phi =$ the angle which the plane PCM makes with the plane of the paper, $\delta r =$ the thickness of the shell, and let ρ denote the density of the shell.

The volume of the elementary solid at M is $r^2 \sin\theta\, \delta r\, \delta\theta\, \delta\phi$ (see Art. 130). The attraction of the whole shell acts along PC; the attraction of the element at M resolved along PC is

$$\frac{\rho r^2 \sin\theta\, \delta r\, \delta\theta\, \delta\phi}{y^2} \cdot \frac{c - r\cos\theta}{y}.$$

We shall eliminate θ from this expression by means of the equation

$$y^2 = c^2 + r^2 - 2rc\cos\theta;$$

therefore

$$\sin\theta \frac{d\theta}{dy} = \frac{y}{cr},$$

$$c - r\cos\theta = \frac{y^2 + c^2 - r^2}{2c}.$$

Therefore the attraction of M on P along PC

$$= \frac{\rho r \delta r}{2c^2}\left(1 + \frac{c^2 - r^2}{y^2}\right) \delta y\, \delta\phi.$$

Hence the resultant attraction of the whole shell

$$= \frac{\rho r\,\delta r}{2c^2}\int_{c-r}^{c+r}\int_0^{2\pi}\left(1+\frac{c^2-r^2}{y^2}\right)dy\,d\phi = \frac{\pi\rho r\delta r}{c^2}\int_{c-r}^{c+r}\left(1+\frac{c^2-r^2}{y^2}\right)dy$$

$$= \frac{\pi\rho r\delta r}{c^2}(2r+2r) = \frac{4\pi\rho r^2\delta r}{c^2} = \frac{\text{mass of the shell}}{c^2}.$$

This result shews that the shell attracts the particle at P in the same manner as if the mass of the shell were condensed at its centre.

212. It follows from the preceding Article, that a sphere which is either homogeneous or consists of concentric spherical shells of uniform density, attracts the particle at P in the same manner as if the whole mass were collected at its centre.

213. *To find the attraction of a homogeneous spherical shell of small thickness on a particle placed within it.*

We must proceed as in Art. 211; but the limits of y are in this case $r-c$ and $r+c$; hence the resultant attraction of the shell

$$= \frac{\pi\rho r\delta r}{c^2}\int_{r-c}^{r+c}\left(1-\frac{r^2-c^2}{y^2}\right)dy = \frac{\pi\rho r\delta r}{c^2}(2c-2c) = 0.$$

Therefore a particle within the shell is equally attracted in every direction.

Suppose a particle inside a homogeneous sphere at the distance r from its centre; then by what has just been shewn all that portion of the sphere which is at a greater distance from the centre than the particle produces no effect on the particle. Also by Art. 211, the remainder of the sphere attracts the particle in the same manner as if the mass of the remainder were all collected at the centre of the sphere. Thus if ρ be the density of the sphere the attraction on the particle is

$$\frac{\frac{4}{3}\pi\rho r^3}{r^2}, \text{ that is } \frac{4\pi\rho r}{3}.$$

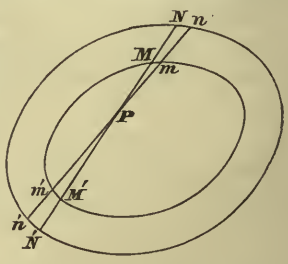
Thus *inside a homogeneous sphere the attraction varies as the distance from the centre.*

214. The propositions respecting the attraction of a uniform spherical shell on an external or internal particle were

given by Newton (*Principia*, Lib. I. Prop. 70, 71). The result with respect to the *internal* particle was extended by Newton to the case of a shell bounded by similar and similarly situated surfaces formed by the revolution of an ellipse round an axis. (*Principia*, Lib. I. Prop. 91, Cor. 3.) The proposition is also true when the shell is bounded by similar and similarly situated *ellipsoidal* surfaces, which we proceed to demonstrate in the method given by Newton for the surfaces of revolution.

215. *If a shell of uniform density be bounded by two ellipsoidal surfaces which are concentric, similar, and similarly situated, the resultant attraction on an internal particle vanishes.*

Let the attracted particle P be the vertex of an infinite series of right cones. Let $NMPM'N'$ and $nmPm'n'$ be two generating lines of one of these cones, and suppose the curves in the figure to represent the intersection of the surfaces of the shell by a plane containing these generating lines. The curves will be similar and similarly situated ellipses, and by a property of such ellipses,



$$MN = M'N' \text{ and } mn = m'n'.$$

By taking the angle of the cone small enough, each of the two portions of the shell which it intercepts will be ultimately a *frustum of a cone*, and being of equal altitude and having a common vertical angle, they *will exercise equal attractions on* P. (See Art. 209.) Similar considerations hold with respect to each of the infinite series of cones of which P is the vertex, and consequently the resultant attraction of the shell vanishes.

This result being true, whatever be the thickness of the shell, is true when the shell becomes indefinitely thin.

216. Let P be a point external to a uniform thin spherical shell, C the centre of the shell, and Q a point in CP such that $CP \,.\, CQ$ = square of radius. If any small double cone with its vertex at Q be described as in Art. 215 cutting two small portions from the shell, it can be shewn without difficulty that these two portions exert equal attractions at

P, and that their attractions make equal angles with the axis PC. Hence if any plane be drawn through Q, the attractions at P of the two parts into which it divides the shell are equal and equally inclined to the axis. It can also be shewn that if ω denote the solid angle of the small cone, each of the two portions cut from the shell has a component attraction along the axis equal to the attraction which a portion of the shell of area $a^2\omega$ would exert if placed at the centre C. Thus each of the two portions into which a plane through Q divides the shell has a component attraction along the axis equal to that which half the shell would exert if collected at the centre.

The next two Articles contain approximate investigations respecting the attraction of spheroids. Two terms will be first defined.

The figure formed by the revolution of an ellipse round its minor axis will be called an *oblatum;* and the figure formed by the revolution of an ellipse round its major axis will be called an *oblongum.* The former figure is often called an oblate spheroid, and the latter a prolate spheroid; but these terms are inconvenient because the word spheroid is used by some writers to denote a figure which is nearly spherical, and by others to denote a figure not limited to any particular form: see page xiii. of the author's *History of the Mathematical Theories of Attraction and of the Figure of the Earth.*

217. *To find the attraction of a homogeneous oblatum of small excentricity on a particle at its pole.*

Let $2c$ be the length of the minor axis and $2a$ that of the major axis of the generating ellipse. The oblatum may be supposed made up of a concentric sphere, the radius of which is c, and an exterior shell; we shall calculate the attractions of these portions separately.

Let a section be made of the sphere and the oblatum by a plane perpendicular to the axis of revolution of the oblatum at a distance x from the attracted particle. This plane cuts the sphere and the oblatum in concentric circles; the area of the former being πy^2 and of the latter $\frac{\pi a^2 y^2}{c^2}$, where $y^2 = 2cx - x^2$;

the difference of these areas is $\pi\left(\frac{a^2}{c^2}-1\right)y^2$. If a section be made by a second plane, parallel to the former and at a distance δx from it, the volume of the portion of the *shell* intercepted between the planes will be $\pi\left(\frac{a^2}{c^2}-1\right)y^2\delta x$. The distance of every particle of the annulus thus formed from the attracted particle is approximately $\sqrt{(2cx)}$, and, as the resultant attraction of the annulus will act along the axis of the oblatum, it will, approximately,

$$= \pi\rho\left(\frac{a^2}{c^2}-1\right)\frac{x}{\sqrt{(2cx)}}\frac{y^2\delta x}{2cx} = \pi\rho\left(\frac{a^2}{c^2}-1\right)\frac{2cx^{\frac{1}{2}}-x^{\frac{3}{2}}}{(2c)^{\frac{3}{2}}}\delta x.$$

Therefore the resultant attraction of the shell

$$= \frac{\pi\rho\,(a^2-c^2)}{2^{\frac{3}{2}}c^{\frac{1}{2}}}\int_0^{2c}(2cx^{\frac{1}{2}}-x^{\frac{3}{2}})\,dx = \frac{8\pi\rho\,(a^2-c^2)}{15c}.$$

If we suppose $c = a\,(1-\epsilon)$, ϵ being very small, we have

$$a^2 - c^2 = 2c^2\epsilon \text{ approximately};$$

therefore the resultant attraction of the shell $= \dfrac{16\pi\rho\epsilon c}{15}$.

Also the attraction of the sphere, by Art. 212, $= \frac{4}{3}\pi\rho c$; therefore the attraction of the oblatum on the particle

$$= \tfrac{4}{3}\pi\rho\,(1+\tfrac{4}{5}\epsilon)\,c.$$

This method is due to Clairaut; see Art. 165 of the *History of the Mathematical Theories of Attraction...*

218. *To find the attraction of a homogeneous oblatum of small excentricity on a particle at its equator.*

Let $2c$ be the length of the minor axis, and $2a$ that of the major axis of the generating ellipse. The oblatum may be supposed to be the difference between a concentric sphere of radius a and a shell, and the attractions of the sphere and shell may be separately calculated. Let a section be made of the sphere and the oblatum by a plane perpendicular to the straight line joining the attracted particle with the common centre of the sphere and oblatum, and at a distance x from the attracted particle; this plane will cut the sphere in a circle the

area of which is πy^2, where $y^2 = 2ax - x^2$, and it will cut the oblatum in an ellipse of which the semi-axes are respectively y and $\frac{cy}{a}$, and the area of which is therefore $\frac{\pi c}{a} y^2$. The difference of the two areas is $\pi \left(1 - \frac{c}{a}\right) y^2$. If a section be made by a second plane parallel to the former, and at a distance δx from it, the volume of the portion of the shell intercepted between the planes will be $\pi \left(1 - \frac{c}{a}\right) y^2 \delta x$. The distance of every particle of the annulus thus formed from the attracted particle is approximately $\sqrt{(2ax)}$; and as the resultant attraction of the annulus will act along the straight line joining the attracted particle with the centre, it will approximately

$$= \pi\rho \left(1 - \frac{c}{a}\right) \frac{x}{\sqrt{(2ax)}} \frac{y^2 \delta x}{2ax} = \pi\rho \left(1 - \frac{c}{a}\right) \frac{2ax^{\frac{1}{2}} - x^{\frac{3}{2}}}{(2a)^{\frac{3}{2}}} \delta x.$$

Therefore the resultant attraction of the shell

$$= \frac{\pi\rho\,(a-c)}{2^{\frac{3}{2}} a^{\frac{5}{2}}} \int_0^{2a} (2ax^{\frac{1}{2}} - x^{\frac{3}{2}})\, dx = \frac{8\pi\rho\,(a-c)}{15}$$

$$= \frac{8\pi\rho a\epsilon}{15}, \text{ if } c = a\,(1-\epsilon).$$

Also the attraction of the sphere, by Art. 212, $= \frac{4}{3}\pi\rho a$; therefore the attraction of the oblatum on the particle

$$= \tfrac{4}{3}\pi\rho a - \frac{8\pi\rho a\epsilon}{15} = \tfrac{4}{3}\pi\rho\,(1 - \tfrac{2}{5}\epsilon)\,a = \tfrac{4}{3}\pi\rho\,(1 + \tfrac{3}{5}\epsilon)\,c.$$

In the same manner it might be shewn that the attractions of a homogeneous *oblongum* of small excentricity on particles at the pole and equator are respectively

$$\tfrac{4}{3}\pi\rho\,(1 - \tfrac{4}{5}\epsilon)\,c \text{ and } \tfrac{4}{3}\pi\rho\,(1 - \tfrac{3}{5}\epsilon)\,c,$$

$2c$ being the axis of revolution of the oblongum, and

$$a = c\,(1 - \epsilon).$$

219. One more example may be given. It is sometimes useful to compare the attraction exerted by the Earth on a

particle at the top of a mountain with the attraction exerted by the Earth on the same particle at the ordinary level of the Earth's surface. The investigation is given by Poisson (*Mécanique*, Tom. I. pp. 492...496). Let r denote the Earth's radius, x the height of the mountain, g the attraction of the Earth on a particle of a unit of mass at the ordinary level of the Earth's surface. If there were no mountain the attraction of the Earth on the particle at a distance x from its surface would be $g\frac{r^2}{(r+x)^2}$: we have then to add to this expression the attraction exerted by the mountain itself. Suppose the mountain to be of uniform density ρ, and consider it to be cylindrical in shape, and the particle to be at the centre of its upper surface; then by Art. 208 the resultant attraction is

$$2\pi\rho\{x - \sqrt{(x^2+b^2)} + b\},$$

where b is the radius of the cylinder. If b is so large in comparison with x that the square of $\frac{x}{b}$ can be neglected, this expression reduces to $2\pi\rho x$. Thus if g' denote the attraction at the top of the mountain

$$g' = \frac{gr^2}{(r+x)^2} + 2\pi\rho x.$$

Let σ denote the *mean* density of the Earth, so that the mass of the Earth is $\frac{4\pi\sigma r^3}{3}$; then

$$g = \frac{4\pi\sigma r^3}{3r^2} = \frac{4\pi\sigma r}{3};$$

thus

$$g' = g\left\{\frac{r^2}{(r+x)^2} + \frac{3\rho x}{2\sigma r}\right\}.$$

This result is substantially due to Bouguer; see Art. 363 of the *History of the Mathematical Theories of Attraction*...

Now the mean density of the Earth is known to be about five and a half times that of water, and from what may be conjectured of the density of matter at the Earth's surface, we may suppose $\frac{\rho}{\sigma} = \frac{1}{2}$. And

$$\frac{r^2}{(r+x)^2} = \left(1+\frac{x}{r}\right)^{-2} = 1 - \frac{2x}{r} \text{ approximately;}$$

thus $$g' = g\left(1 - \frac{2x}{r} + \frac{3x}{4r}\right) = g\left(1 - \frac{5x}{4r}\right).$$

[The assumptions made in this investigation are more applicable to the attraction of elevated table land than of a mountain. The formula obtained is the accepted formula for estimating the difference in the value of g at two places so far as dependent on their heights above sea level.]

We have hitherto confined ourselves to simple examples of the ordinary law of attraction; we now proceed to consider some other laws of attraction, and also some more complex cases of the ordinary law.

220. *If the particles of a body attract with a force varying as the product of the mass into the distance, the resultant attraction of the body is the same as if the whole mass of the body were collected at its centre of gravity.*

Take the centre of gravity of the attracting body as the origin of co-ordinates, and let a, b, c be the co-ordinates of the attracted particle. Divide the attracting body into indefinitely small elements; let x, y, z be the co-ordinates of an element, m its mass, and r its distance from the attracted particle. Then the attraction of this element is mr, and by resolving it parallel to the co-ordinate axes, we obtain

$$mr.\frac{a-x}{r},\ mr.\frac{b-y}{r},\ mr.\frac{c-z}{r},$$

respectively. Hence, if X, Y, Z denote the resolved parts of the whole attraction, we have

$$X = \Sigma m(a-x),\ Y = \Sigma m(b-y),\ Z = \Sigma m(c-z).$$

But, since the origin is the centre of gravity of the attracting body, we have

$$\Sigma mx = 0,\ \Sigma my = 0,\ \Sigma mz = 0;$$

therefore $$X = a\Sigma m,\ Y = b\Sigma m,\ Z = c\Sigma m.$$

But these expressions are the resolved attractions of a mass Σm placed at the origin, which establishes the proposition.

221. *To find the attraction of a homogeneous spherical shell on a particle without it; the law of attraction being represented by $\phi(y)$, where y is the distance.*

If we proceed as in Art. 211, we find the resultant attraction of the shell on P along PC

$$= \frac{\pi\rho r\delta r}{c^2}\int_{c-r}^{c+r}(y^2+c^2-r^2)\,\phi(y)\,dy.$$

Suppose $\int\phi(y)\,dy = \phi_1(y)$, and $\int y\phi_1(y)\,dy = \psi(y)$.

Then, integrating by parts, we have

$$\int(y^2+c^2-r^2)\,\phi(y)\,dy = (y^2+c^2-r^2)\,\phi_1(y) - 2\int y\phi_1(y)\,dy$$

$$= (y^2+c^2-r^2)\,\phi_1(y) - 2\psi(y);$$

therefore

$$\frac{\pi\rho r\delta r}{c^2}\int_{c-r}^{c+r}(y^2+c^2-r^2)\,\phi(y)\,dy$$

$$= 2\pi\rho r\delta r\left\{\frac{c+r}{c}\phi_1(c+r) - \frac{c-r}{c}\phi_1(c-r) - \frac{1}{c^2}\psi(c+r) + \frac{1}{c^2}\psi(c-r)\right\}$$

$$= 2\pi\rho r\delta r\,\frac{d}{dc}\left\{\frac{\psi(c+r)-\psi(c-r)}{c}\right\}.$$

This last form is introduced merely as an analytical artifice to simplify the expression.

222. *To find the attraction of the shell on an internal particle.*

The calculation is the same as in the last Article, except that the limits of y are $r-c$ and $r+c$. Hence, the attraction of the shell

$$= 2\pi\rho r\delta r\left\{\frac{r+c}{c}\phi_1(r+c) + \frac{r-c}{c}\phi_1(r-c) - \frac{1}{c^2}\psi(r+c) + \frac{1}{c^2}\psi(r-c)\right\}$$

$$= 2\pi\rho r\delta r\,\frac{d}{dc}\left\{\frac{\psi(r+c)-\psi(r-c)}{c}\right\}.$$

223. The formulæ of the preceding two Articles will give the attraction when the law of attraction is known.

Example 1. Let $\phi(r) = \frac{1}{r^2}$; therefore $\phi_1(r) = -\frac{1}{r} + A$,

$$\psi(r) = -r + \tfrac{1}{2}Ar^2 + B;$$

A and B being constants.

Therefore the attraction on an external particle

$$= 2\pi\rho r\delta r \frac{d}{dc}\left\{\frac{-4r + A(c+r)^2 - A(c-r)^2}{2c}\right\}$$

$$= 2\pi\rho r\delta r \frac{d}{dc}\left(-\frac{2r}{c} + 2Ar\right) = \frac{4\pi\rho r^2\delta r}{c^2}, \quad \text{(Art. 211.)}$$

The attraction on an internal particle

$$= 2\pi\rho r\delta r \frac{d}{dc}\left\{\frac{-4c + A(r+c)^2 - A(r-c)^2}{2c}\right\}$$

$$= 2\pi\rho r\delta r \frac{d}{dc}\{-2 + 2Ar\} = 0, \text{ (Art. 213).}$$

Example 2. Let $\phi(r) = r$;

therefore $\phi_1(r) = \frac{1}{2}r^2 + A$, $\psi(r) = \frac{1}{8}r^4 + \frac{1}{2}Ar^2 + B$.

The attraction on an external particle

$$= 2\pi\rho r\delta r \frac{d}{dc}\left\{\frac{(c+r)^4 - (c-r)^4 + 4A(c+r)^2 - 4A(c-r)^2}{8c}\right\}$$

$$= 2\pi\rho r\delta r \frac{d}{dc}\{c^3r + r^3 + 2Ar\}$$

$$= 4\pi\rho r^2 c\delta r = \text{mass} \times c. \quad \text{(Art. 220.)}$$

For these two cases, then, the attraction is the same as if the shell were collected at its centre. We shall now ascertain whether there are any other laws which give the same property.

224. *To find what laws of attraction allow us to suppose a spherical shell condensed into its centre when attracting an external particle.*

Let $\phi(r)$ be the law of force; then, if c be the distance of the centre of the shell from the attracted particle, r the

radius of the shell, and $\psi(r) = \int\{r \int \phi(r)\, dr\}\, dr$, the attraction of the shell

$$= 2\pi\rho r \delta r \frac{d}{dc}\left\{\frac{\psi(c+r) - \psi(c-r)}{c}\right\}.$$

But if the shell be condensed into its centre, the attraction

$$= 4\pi\rho r^2 \delta r \phi(c);$$

therefore
$$\frac{d}{dc}\left\{\frac{\psi(c+r) - \psi(c-r)}{c}\right\} = 2r\phi(c).$$

Expand $\psi(c+r)$ and $\psi(c-r)$ in powers of r; then using $\psi'(c)$ for $\frac{d\psi(c)}{dc}$, &c., we have

$$2r\phi(c) = 2\frac{d}{dc}\left\{\frac{r}{c}\psi'(c) + \frac{r^3}{c\lfloor 3}\psi'''(c) + \ldots\right\}$$

$$= 2r\phi(c) + 2\frac{d}{dc}\left\{\frac{r^3}{c\lfloor 3}\psi'''(c) + \ldots\right\};$$

therefore
$$\frac{d}{dc}\left\{\frac{r^3}{c\lfloor 3}\psi'''(c) + \ldots\right\} = 0,$$

whatever r may be; therefore

$$\frac{d}{dc}\left\{\frac{\psi'''(c)}{c}\right\} = 0, \quad \frac{d}{dc}\left\{\frac{\psi''''(c)}{c}\right\} = 0, \text{ \&c.}$$

But
$$\psi'(c) = c\int\phi(c)\, dc;$$

therefore
$$\psi''(c) = \int\phi(c)\, dc + c\phi(c);$$

therefore
$$\psi'''(c) = 2\phi(c) + c\phi'(c).$$

Therefore, by the first of the above equations of condition for $\psi(c)$,

$$\frac{2\phi(c)}{c} + \phi'(c) = \text{a constant}.$$

Put $3A$ for this constant; multiply both sides of the equation by c^2 and integrate; thus

$$c^2\phi(c) = Ac^3 + B;$$

therefore
$$\phi(c) = Ac + \frac{B}{c^2}.$$

This value satisfies all the other equations of condition for $\psi(c)$; therefore the required laws of attraction are those of the direct distance, the inverse square, and a law compounded of these.

See also Art. 1046 of the *History of the Mathematical Theories of Attraction...*

225. *To find for what laws the shell attracts an internal particle equally in every direction.*

When this is the case,

$$\frac{d}{dc}\left\{\frac{\psi(r+c)-\psi(r-c)}{c}\right\}=0;$$

therefore
$$\psi'(r)+\frac{c^2}{\lfloor 3}\psi'''(r)+\ldots=A,$$

whatever c is, A being a constant independent of c; therefore

$$\psi'(r)=A,\quad \psi'''(r)=0, \text{ \&c.}$$

From the second condition, we have

$$\psi(r)=B+B'r+B''r^2,$$

where B, B', and B'' are constants.

Hence
$$\psi'(r) \text{ or } r\int\phi(r)\,dr=B'+2B''r;$$

therefore
$$\int\phi(r)\,dr=\frac{B'}{r}+2B'';$$

therefore
$$\phi(r)=-\frac{B'}{r^2}:$$

with this value of $\phi(r)$ all the other equations of condition are satisfied: hence the only law which satisfies the condition is that of the inverse square.

226. *To prove that the resultant attraction of the particles of a finite body of any figure on a particle of which the distance is very great in comparison with the greatest diameter of the attracting body, is very nearly the same, as if the particles were condensed at their centre of gravity and attracted according to the same law, whatever that law be.*

Let the origin of co-ordinates be taken at the centre of gravity of the attracting body, the axis of x through the attracted particle; let c be its abscissa, and x, y, z the co-ordinates of any particle of the body, ρ the density of that particle.

Then the distance between these two particles, or r,

$$= \sqrt{\{(c-x)^2 + y^2 + z^2\}}.$$

Let $r\phi(r^2)$ be the law of attraction; then the whole attraction parallel to the axis of x

$$= \iiint \rho\,(c-x)\,\phi\,(c^2 - 2cx + x^2 + y^2 + z^2)\,dx\,dy\,dz,$$

the limits being obtained from the equation to the surface of the body. This attraction therefore

$$= \iiint \rho\,(c-x)\,\{\phi\,(c^2) - (2cx - x^2 - y^2 - z^2)\,\phi'\,(c^2) + \ldots\}\,dx\,dy\,dz$$

$$= c\phi(c^2)\iiint\rho\left\{1 - \frac{x}{c}\left(1 + \frac{2c^2\phi'(c^2)}{\phi\,(c^2)}\right) + (y^2+z^2+3x^2)\frac{\phi'(c^2)}{\phi\,(c^2)} + \ldots\right\}dx\,dy\,dz$$

$$= Mc\phi\,(c^2) + c^3\phi'(c^2)\iiint\rho\,\frac{y^2+z^2+3x^2}{c^2}\,dx\,dy\,dz + \ldots\ldots\ldots(A),$$

M being the mass of the body, and $\iiint\rho x\,dx\,dy\,dz = 0$, since x is measured from the centre of gravity of the body.

Now suppose x, y, z to be exceedingly small in comparison with c; then all the terms of (A) after the first are extremely small in comparison with that term, it being observed that $c^3\phi'\,(c^2)$ is of the same order as $c\phi\,(c^2)$ in terms of c. Hence the resultant attraction is very nearly $Mc\,\phi\,(c^2)$; that is, it is very nearly the same as if the particles were condensed at their centre of gravity and attracted according to the law determined by the function $r\phi\,(r^2)$.

227. From Art. 224, it appears that when the law of attraction is that of the inverse square of the distance, a sphere composed of shells, each of which is homogeneous, attracts an external particle with a resultant force, which is the same as if the sphere were condensed at its centre. It

may be shewn also that *two such spheres* attract each other in the same manner as if each were condensed at its centre. For consider any element of mass forming part of the first sphere; the attraction of this on the second sphere will be equal and opposite to the resultant attraction of the second sphere on it, and will therefore be the same as if the second sphere were collected at its centre. Similarly, the attraction of any other element of the first sphere on the second will be the same as if the second were collected at its centre. Proceeding thus, we find that the whole action of the first sphere on the second is the same as if the second were collected at its centre, and therefore the mutual attraction of the spheres is the same as if each were collected at its centre.

If the law of attraction be that of the direct distance, then two bodies of any shape attract each other with a resultant force which is the same as if each were collected at its centre of gravity.

In the following Articles 228—240 we shall establish by special methods some results of great practical importance.

228. The following proposition is due to Gauss (Taylor's *Scientific Mem.* Vol. III. pp. 179—181). Let N denote the normal attraction at any point of a closed surface, that is, the attraction resolved normally to the surface and considered positive if directed inwards, and let dS be the area of an indefinitely small element of the surface; then *the integral of* NdS *taken over the whole surface is equal to* $4\pi M$, *where* M *denotes the whole quantity of matter within the surface.*

First let the attraction be due to a single particle, of mass m, situated in the interior. Consider the elementary cone which has dS for its base and has its vertex at the attracting particle. If r be the length of the cone and $d\omega$ its solid angle, its base is $r^2 d\omega \sec \phi$, where ϕ denotes the angle which the base makes with a normal section of the cone. Also $\frac{m}{r^2}$ is the whole attraction and $\frac{m}{r^2}\cos\phi$ the normal com-

ponent. Thus we have

$$dS = r^2 d\omega \sec \phi, \quad N = \frac{m}{r^2} \cos \phi,$$

whence $$NdS = md\omega.$$

Integrating over the whole surface, we have

$$\int N ds = m \int d\omega = 4\pi m;$$

hence the proposition is proved for a single particle.

If the surface has folds and is cut in more points than one by a line drawn from the attracting particle, the number of points of meeting will be odd, since at the first cutting we pass from inside to outside, at the second from outside to inside, and so on alternately. The elementary cone will accordingly have an odd number of sections, which we will denote in order by dS_1, dS_2, &c. with corresponding notation for N; then the odd elements dS_1, dS_3, &c. will have their inner faces turned towards the particle, and the even elements dS_2, dS_4, &c. their outer faces. The areas dS_1, dS_2, &c. are to be regarded as essentially positive, but N_2, N_4, &c. will be negative as being directed outwards. Hence we shall have

$$N_1 dS_1 = -N_2 dS_2 = N_3 dS_3 = -N_4 dS_4 = N_5 dS_5 = md\omega,$$

and $$N_1 dS_1 + N_2 dS_2 + N_3 dS_3 + N_4 dS_4 + N_5 dS_5 = md\omega.$$

Next let the attraction be due to a particle of mass m situated without the surface. Every line drawn from it will meet the surface in an even number of points; and with notation similar to the above we shall have

$$-N_1 dS_1 = N_2 dS_2 = -N_3 dS_3 = N_4 dS_4 = md\omega,$$

$$N_1 dS_1 + N_2 dS_2 + N_3 dS_3 + N_4 dS_4 = 0.$$

Hence the integral of NdS over the whole surface vanishes.

Lastly let the attracting matter consist of masses m_1, m_2, m_3, &c. within the surface, and any other particles without the surface. The whole value of N at any point of the surface is the sum of its values for the separate particles. Hence the whole value of the integral $\int NdS$ will be the sum of the partial values of $\int NdS$ obtained by considering each particle

separately. These partial values are, as above proved, $4\pi m_1$, $4\pi m_2$, &c. for the internal particles, and zero for each external particle. We have therefore, on the whole,

$$\int N dS = 4\pi\ (m_1 + m_2 + m_3 + \ldots) = 4\pi M,$$

where M denotes the whole included mass.

The following are examples of the application of this theorem.

229. To find the attraction of a sphere or spherical shell composed of matter symmetrically distributed round the centre.

By symmetry, the attraction is the same at all points of a concentric spherical surface, and is in the direction of a radius drawn inwards. Let r be the radius of any such surface; then we have

$$\int N dS = N \int dS = N 4\pi r^2.$$

Equating this to $4\pi M$, we have

$$N = \frac{M}{r^2}.$$

N in this case is the whole attraction at a point at distance r from the centre, and the result shews that it is the same as the attraction of a mass M collected at the centre. If r is greater than or equal to the radius of the sphere or the external radius of the shell, M will be the whole mass of the sphere or shell. If r is equal to or less than the internal radius of the shell, M is zero.

230. To find the attraction of a cylinder, either solid or hollow, of indefinitely great length, the density being a function of the distance from the axis.

By symmetry, the attraction is the same at all points of a cylindric surface coaxal with the given cylinder, and is directed normally inwards. Let such a cylinder of radius r be cut by two planes at unit distance apart, perpendicular to the axis, and let us take the integral of $N dS$ over the surface of the right cylinder of unit length thus enclosed. The value of N at any point of either end of this cylinder is zero, since

the whole force is tangential; whereas at any point of the convex surface the whole force is normal.

Hence if F denote the attraction at distance r from the axis, we have

$$\int N dS = F \times \text{convex surface} = 2\pi r F.$$

Equating this to $4\pi M$, we have $F = \frac{2M}{r}$.

If r is greater than or equal to the external radius R of the cylinder, M is the mass of unit length of the given cylinder. If the cylinder is hollow, and r is less than its internal radius, M is zero.

231. To find the attraction of a uniform lamina bounded by two parallel planes extending to an indefinite distance in all directions.

By symmetry, the attraction will be normal to the lamina, and will be the same at all points equidistant from the lamina whether on the same or on opposite sides.

Consider a right cylinder with plane ends, these latter being parallel to the lamina and at equal distances on opposite sides, and integrate NdS over its surface. N will be zero over the convex surface because the attraction there is tangential. On the plane ends, which we will suppose to be of unit area, the attraction will have a uniform value which we will call F.

Then $\int NdS$ is $2F$ and is to be equated to $4\pi M$, giving

$$F = 2\pi M,$$

where M is the mass of unit area of the lamina.

To find the attraction of a lamina of indefinite extent with density a function of distance from one of the two plane faces, we may divide it into indefinitely thin laminæ and apply the above result to each. Thus the attraction at an internal point will be

$$2\pi (M_1 - M_2),$$

where M_1 and M_2 denote the masses on opposite sides of a plane drawn through the attracted point, the area of each being unity.

232. We now proceed to explain the subject of *Potential*, which occupies an important place in the Theory of Attraction.

Let m be the mass of a particle attracting according to the law of nature, so that the force which it exerts upon a particle of unit mass at distance r is $\frac{m}{r^2}$. Let the attracted particle move through an elementary arc ds, and let dr denote the corresponding change in r. Then the work done by the attracting force is $-\frac{m}{r^2}dr$ or $d\frac{m}{r}$. Hence if the attracted particle moves by any path from a distance r to a distance r' the work done is $\frac{m}{r'}-\frac{m}{r}$, and if it moves from infinite distance to distance r, the work done is $\frac{m}{r}$. This quantity $\frac{m}{r}$ is called the *potential due to the mass m at the distance r*. It may be defined as *the work done in attracting unit mass from infinity to the specified point*. The student will note that the work done is the same along all paths that join two given extreme points.

Next let there be any number of attracting particles, their masses being m_1, m_2... and their distances from the attracted particle (still supposed to be of unit mass) being r_1, r_2... The above proof shews that the work done in an indefinitely small displacement of the attracted particle is $d\left(\frac{m_1}{r_1}+\frac{m_2}{r_2}+\ldots\right)$ which may be written $d\Sigma\frac{m}{r}$, that the work done when the particle moves into a new position in which its distances from the attracting particles are r_1', r_2'... is

$$\left(\frac{m_1}{r_1'}+\frac{m_2}{r_2'}+\ldots\right)-\left(\frac{m_1}{r_1}+\frac{m_2}{r_2}+\ldots\right)$$

which may be written $\Sigma\frac{m}{r'}-\Sigma\frac{m}{r}$, and that the work from infinity to the original position would be $\Sigma\frac{m}{r}$. The

quantity $\Sigma \frac{m}{r}$ is called the *potential* at the specified point due to all the attracting particles considered, and may be defined as the work which their attraction would do upon a particle of unit mass moving from infinite distance to the specified point. We shall denote $\Sigma \frac{m}{r}$ by V, that is to say the value of V for any point will denote the potential at this point due to the attracting matter which is considered. The work done by the attracting forces when the particle moves from a point where the potential is V to a point where it is V' is $V' - V$; or to state the same thing briefly in words *work done is equal to increase of potential.*

233. The work done in moving through an elementary arc δs will be $\frac{dV}{ds}\delta s$. Hence $\frac{dV}{ds}$ is the effective component of the force acting on the unit particle. In other words the intensity of the component force in any direction at a specified point is the rate at which V increases per unit distance measured in that direction. Since the component force in any direction is proportional to the cosine of the angle which this direction makes with the resultant force, V will remain unchanged when we travel in a direction perpendicular to the resultant force, and the locus of the points for which V has a given value will be a surface over which the resultant force is everywhere normal. Such surfaces are called *equipotential surfaces,* or *surfaces of equilibrium.*

234. If we start from any point and travel always in the direction of the resultant force of attraction, we trace out a *line of force.* A line of force may be defined as *a line to which the resultant force is everywhere tangential.* From what has been just said it is obvious that *lines of force cut equipotential surfaces orthogonally.* The whole of any space pervaded by attraction may be regarded as traversed by an indefinite number of lines of force which are cut orthogonally by an indefinite number of equipotential surfaces.

235. Consider two consecutive equipotential surfaces. Let the potential over one of them be V, and over the other

$V + \delta V$; then the work from any point of the former to any point of the latter is δV, and if δn denote the distance between them at any part, measured along a line of force, the force at this part will be $\frac{\delta V}{\delta n}$, which varies inversely as δn. Thus *the force at different parts of the space intervening between two given consecutive equipotential surfaces varies inversely as the distance between the surfaces.*

Again, consider any small tube bounded by lines of force and thus separating one small bundle of lines of force from the rest. Such a tube is called a *tube of force* or a *force-tube.* If we apply to a portion of it bounded by two normal sections (which will be elements of two equipotential surfaces) the theorem

$$\int N dS = 4\pi M,$$

N will vanish over the sides of the tube because the force there is tangential; at one end (that for which V is least) N will be positive and equal to the resultant force, which we will denote by F; at the other it will be negative and equal to $-F'$ if F' denote the resultant force at this end. Hence, denoting the areas of the two sections by dS, dS', we have

$$FdS - F'dS' = 4\pi M,$$

where M denotes the mass of attracting matter contained within the portion of the tube considered.

When there is no attracting matter within the tube, we have

$$FdS = F'dS',$$

that is *the force varies inversely as the cross-section of the tube.* The student can verify this result by considering the forms of the tubes of force when the attracting body is a sphere, an indefinite cylinder, or an indefinite lamina.

236. It is easy to find the forms of the lines of force and equipotential surfaces when the attracting body is a uniform straight line. For we have seen in Art. 204 that if AB be the line and P the attracted point, the direction of the force bisects the angle APB. Hence every hyperbola with A and B as foci will be a line of force, and every ellipsoid of revolution with A and B as foci will be an equipotential surface.

237. Potential may have a constant finite value not only over a surface but throughout a portion of space in three dimensions. We have an example of this in the case of a spherical shell. There is no force in the hollow space within it, hence $\frac{dV}{ds}$ must vanish for all directions of ds at any point of this space, that is V must be constant; and that it is not zero is obvious from the fact that it is the sum of such terms as $\frac{m}{r}$, each of which is positive.

238. The attraction of a sphere or spherical shell at an external point is the same as if all the mass were collected at the centre. Hence the work done by this attraction on a particle coming from infinite distance to an external point must be the same. Therefore the potential due to a sphere or spherical shell at any external point is the mass of the sphere or shell divided by the distance of the external point from the centre.

239. The mean value of V over the whole of a spherical surface is equal to the value of V at the centre, provided that none of the attracting matter lies within the surface. For let m be the mass of one of the attracting particles, dS an element of the spherical surface, and r the distance of m from dS. Then the potential due to m at any point of the surface is $\frac{m}{r}$, and the mean value of this over the surface is the quotient of $\int \frac{m}{r}\,dS$ by $\int dS$; but $\int \frac{m}{r}\,dS$ is equal to the potential which would be produced at the point where m is situated by a thin spherical shell whose mass per unit area is m, since $m\,dS$ would be the expression for the mass of an element of such a shell; and we know that this is the same as if the whole mass of the shell were collected at the centre. Let a denote the radius of the sphere and c the distance of its centre from m, then we have

$$\frac{\int \frac{m}{r}\,dS}{\int dS} = \frac{\frac{4\pi a^2 m}{c}}{4\pi a^2} = \frac{m}{c};$$

but $\frac{m}{c}$ is the potential at the centre due to m. Hence the theorem is true for the potential due to a single particle, and since the potential due to a whole is the sum of the potentials due to the parts, the theorem is true universally.

Hence we can easily prove that if V has a constant value V_0 throughout a given finite region, it must have this same value in all the space that could be reached from this region without passing through attracting matter. For otherwise we could describe a sphere having its centre and part of its surface within the region where V has the constant value V_0, while the rest of its surface is in a region where V has a value everywhere greater or everywhere less than V_0, and the mean value of V over the surface would be different from the value at the centre.

In a similar manner it can be proved that potential cannot have a maximum or minimum value at a point in empty space.

240. When a liquid flows through a tube keeping it always full, the flow of liquid (that is, the volume of liquid that passes per unit time) must be the same for all sections of the tube,—not only for all normal sections, but for oblique sections also. If dS be any section, and N the component velocity of the liquid normal to dS, the flow across the section will be NdS. If any section be made of a liquid in motion, the flow from one side to the other of this section will be $\int NdS$, dS denoting an element of the section, and N the velocity of the liquid which passes dS resolved normally to dS. If we consider any closed surface in the liquid, the flow into it must equal the flow out of it, and therefore if N be reckoned positive or negative according as the flow is inward or outward, the integral $\int NdS$ over the whole closed surface will be zero.

From this analogy, the product NdS where N denotes the intensity of the component force normal to the element dS, is sometimes called the *flow* or *flux of force* across the element. We can then assert that the flow of force is the same across any two sections of a tube of force not separated

by attracting matter, that the total flow of force into a space not containing attracting matter is zero, and that the total flow of force into a space containing the quantity M of attracting matter is $4\pi M$.

Faraday used the expression "number of lines of force" to denote what is here called flow of force; and it has sometimes been called "number of unit tubes of force." The student will therefore note that "the number of unit tubes of force which cut" a surface, the "flux of force across" the surface, and the "integral of normal attraction over" the surface are various names for the integral $\int N dS$.

The three following Articles contain a discussion of Potential from a different point of view and furnish a confirmation of some of the foregoing results.

241. Let there be a body of any form; let ρ represent the density of an element, the volume of which is $dadbdc$, a, b, c being the co-ordinates of the element. Suppose the attraction between two particles of masses m and m' respectively, at a distance r, to be $mm' F(r)$; then the components X, Y, Z in the directions of the positive axes of x, y, z, of the attraction of the body on a particle whose mass is unity, and co-ordinates x, y, z are found by the equations

$$X = \iiint \rho \frac{a-x}{r} F(r)\, dadbdc,$$

$$Y = \iiint \rho \frac{b-y}{r} F(r)\, dadbdc,$$

$$Z = \iiint \rho \frac{c-z}{r} F(r)\, dadbdc,$$

r being $= \{(x-a)^2 + (y-b)^2 + (z-c)^2\}^{\frac{1}{2}}$.

The integrations are to be taken so as to include all the elements of the attracting body.

Let $\phi(r)$ be such a function of r that $F(r)$ is its differential coefficient with respect to r, and let

$$U = \iiint \rho\phi(r)\, dadbdc,$$

the integrations being extended so as to include all the ele-

ments of the attracting body; then will

$$X = -\frac{dU}{dx}, \quad Y = -\frac{dU}{dy}, \quad Z = -\frac{dU}{dz}.$$

For $\frac{d\phi(r)}{dx} = \frac{d\phi(r)}{dr}\frac{dr}{dx} = F(r)\frac{dr}{dx} = -F(r)\frac{a-x}{r}$;

therefore

$$X = -\iiint \rho \frac{d\phi(r)}{dx}\, dadbdc$$

$$= -\frac{d}{dx}\iiint \rho\phi(r)\, dadbdc$$

$$= -\frac{dU}{dx}.$$

Similarly, the equations $Y = -\frac{dU}{dy}$ and $Z = -\frac{dU}{dz}$ may be established.

It may be observed that if in any case, for example that of an infinite solid, the integral U becomes infinite, but the differential coefficients $\frac{dU}{dx}$, $\frac{dU}{dy}$, $\frac{dU}{dz}$ are finite, the preceding values of X, Y, Z will still be correct.

For suppose we take a finite portion of the solid; the components of its attraction will have for values the differential coefficients of U. Suppose now that we extend without limit the portion of the mass considered, the components of the attraction will always be

$$-\frac{dU}{dx}, -\frac{dU}{dy}, -\frac{dU}{dz},$$

whether U increase without limit or not. Hence, if these three expressions tend to limits, those limits will be the components of the attraction of the infinite solid. And if they increase indefinitely, we may conclude that the attraction increases without limit as the portion of the body considered increases; this we express by saying that the attraction of the solid is infinite.

242. If the law of attraction be that of the inverse square, we have

$$F(r) = \frac{1}{r^2}, \text{ and } \phi(r) = -\frac{1}{r}.$$

Let $V = -U$, that is, let

$$V = \iiint \frac{\rho da db dc}{r} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1);$$

then, by the preceding Article, we have for the attractions in the directions of the positive axes of x, y, z respectively,

$$X = \frac{dV}{dx}, \quad Y = \frac{dV}{dy}, \quad Z = \frac{dV}{dz}.$$

The equation which gives V is equivalent to the following operation:—*decompose the attracting mass into indefinitely small elements, and divide the mass of each element by the distance of that element from the attracted particle; the sum of these quotients is* V. Hence, the value of V will be quite independent of the axes, rectangular or polar, which we may find it convenient to employ. Suppose we use the ordinary polar formulæ, and take the position of the attracted particle for the origin; then the element of volume is (Art. 130) $r^2 \sin\theta \, \delta\phi \, \delta\theta \, \delta r$; therefore

$$V = \iiint \rho r \sin\theta \, d\phi \, d\theta \, dr \ldots\ldots\ldots\ldots (2).$$

Suppose the attracted particle forms part of the attracting mass; then, since r vanishes for those particles of the attracting mass which are in contact with the attracted particle, from equation (1) it would be doubtful if V is *finite* in this case; but from (2) we see that it really is finite.

243. *To express by means of* V *the attraction resolved along any line.*

Let s be the length of the arc of any curve measured from a fixed point up to P the attracted particle; l, m, n the direction cosines of the tangent to this line at P; R the attraction resolved along this tangent; then

$$R = lX + mY + nZ$$
$$= l\frac{dV}{dx} + m\frac{dV}{dy} + n\frac{dV}{dz}.$$

Now, if we restrict ourselves to points lying on the line s, V will become a function of s alone; for V is a function of x, y, and z, and each of these may be regarded as a function of s; thus we shall have by the differential calculus,

$$\frac{dV}{ds} = \frac{dV}{dx}\frac{dx}{ds} + \frac{dV}{dy}\frac{dy}{ds} + \frac{dV}{dz}\frac{dz}{ds};$$

and since $\frac{dx}{ds} = l$, $\frac{dy}{ds} = m$, $\frac{dz}{ds} = n$, we get

$$R = \frac{dV}{ds}.$$

244. *To calculate the value of V in the case of a spherical shell, the density being a function of the distance from the centre.*

Let a be the distance of P from the centre; u the distance of any point in the attracting shell from the centre; θ and ϕ the other polar co-ordinates of this point; let θ be measured from the straight line joining the centre of the sphere with the attracted particle P, which is obviously the direction of the resultant attraction. The mass of the element at this point is $\rho u^2 \sin\theta\, \delta u\, d\theta\, \delta\phi$, and

$$V = \int_{u_1}^{u_2}\int_0^{\pi}\int_0^{2\pi} \frac{\rho u^2 \sin\theta\, du\, d\theta\, d\phi}{r},$$

where u_1 and u_2 are the internal and external radii of the shell; hence

$$V = 2\pi \int_{u_1}^{u_2}\int_0^{\pi} \frac{\rho u^2 \sin\theta\, du\, d\theta}{r}.$$

Now $$r^2 = u^2 - 2au\cos\theta + a^2;$$

therefore $$\sin\theta \frac{d\theta}{dr} = \frac{r}{au},$$

and $$V = \frac{2\pi}{a}\iint \rho u\, du\, dr.$$

We must now distinguish three cases.

I. When P is *beyond* the external surface, the limits of r are $a-u$ and $a+u$; therefore

$$V = \frac{2\pi}{a}\int_{u_1}^{u_2}\int_{a-u}^{a+u} \rho u \, du \, dr$$

$$= \frac{4\pi}{a}\int_{u_1}^{u_2} \rho u^2 du \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

But if M denote the mass of the spherical shell,

$$M = 4\pi \int_{u_1}^{u_2} \rho u^2 du;$$

therefore

$$V = \frac{M}{a}.$$

Now the resultant attraction $= \frac{dV}{da} = -\frac{M}{a^2}$, so that it is the same as if the mass of the shell were collected at its centre.

II. When P is *within* the internal surface, the limits of r are $u-a$ and $u+a$; therefore

$$V = \frac{2\pi}{a}\int_{u_1}^{u_2}\int_{u-a}^{u+a} \rho u \, du \, dr$$

$$= 4\pi \int_{u_1}^{u_2} \rho u \, du \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Since this is independent of a, we have

$$\frac{dV}{da} = 0.$$

That is, the attraction vanishes.

III. By combining the results contained in equations (1) and (2), we see that if P be *between* the bounding surfaces of the shell,

$$V = \frac{4\pi}{a}\int_{u_1}^{a} \rho u^2 du + 4\pi \int_{a}^{u_2} \rho u \, du.$$

That is, the *resultant attraction is the same as if all the matter which is nearer to the centre than P were collected at the centre, and the rest of the matter neglected.*

245. Attraction and repulsion varying inversely as the square of the distance form the bases of the mathematical theories both of Electrostatics and of Magnetism. The formulæ which we have hitherto employed can be extended to repulsion by assigning negative values to the attracting masses. If $\frac{m}{r^2}$ denotes an attraction, $\frac{-m}{r^2}$ will denote a repulsion. We shall give one example, taken from Magnetism.

The simplest imaginable magnet has two points A and B (called its poles) such that the action of the magnet at an external point P is the resultant of two forces one directed from A to P and the other from P to B, their intensities being respectively $\frac{m}{AP^2}$ and $\frac{m}{BP^2}$, where m is a quantity independent of the position of P.

We shall investigate the equation to the lines of force in terms of the distances $AP = r$ and $BP = r'$.

Let Q be a point very near to P on the line of force through P, so that PQ may be taken as the direction of the resultant force at P; then the sines of the angles which the resultant makes with the two components are inversely as the components, and therefore, in the present case, are directly as the squares of the distances. Thus we have

$$r^2 : r'^2 :: \sin QPA : \sin QPB.$$

But by joining AQ, BQ and dropping perpendiculars from Q on AP and BP we can shew that

$$QP \sin QPA = -rdA, \text{ and } QP \sin QPB = r'dB,$$

where A denotes the angle BAP and B the angle ABP.

Thus we have

$$r^2 : r'^2 :: -rdA : r'dB,$$

whence,

$$rdB + r'dA = 0.$$

But $r : r' :: \sin B : \sin A$, therefore $\sin BdB + \sin AdA = 0$, whence, by integration

$$\cos A + \cos B = C,$$

where C is a quantity which is constant for any one line of force but varies from one line to another.

It can be shewn that if any point R be taken in the production of AB, the locus of the points of contact of tangents drawn from R to the magnetic lines of force is a circle.

If the law had been that of the direct distance, similar reasoning would have led to the result $A + B =$ constant, shewing that the lines of force are circles passing through A and B.

246. Since the potential at P (with the law of nature) is $-\frac{m}{r}+\frac{m}{r'}$, the equation to an equipotential surface is

$$\frac{1}{r'}-\frac{1}{r}=C,$$

where C is a quantity which is constant for any one surface but varies from one surface to another.

The case in which the distance AB, which we shall denote by l, is very small compared with AP or BP is of special importance. Let θ denote the angle ABP, then $r'-r$ will be $l\cos\theta$, and the potential at P will be

$$V=\frac{m}{r'}-\frac{m}{r}=\frac{m(r-r')}{rr'}=-\frac{ml\cos\theta}{r^2}=-\mu\frac{\cos\theta}{r^2}$$

if we put μ for the constant quantity ml (called the moment of the magnet).

The intensity of the component force along AP or BP is

$$\frac{dV}{dr}=\mu\frac{2\cos\theta}{r^3},$$

and the intensity of the component perpendicular to this (reckoned positive if in the direction of increasing θ) is

$$\frac{dV}{rd\theta}=\mu\frac{\sin\theta}{r^3}.$$

Hence the whole force has the intensity $\frac{\mu}{r^3}\sqrt{\sin^2\theta+4\cos^2\theta}$

or $\frac{\mu}{r^3}\sqrt{1+3\cos^2\theta}$, and makes an angle ϕ with the production of AP or BP such that $\tan\phi = \frac{1}{2}\tan\theta$.

The actual direction and intensity of terrestrial magnetic force approximate (very roughly) to these results, the small magnet AB being at the centre of the earth and directed towards the magnetic poles.

To deduce the equation to the lines of force, we have, from above,

$$\frac{dr}{rd\theta} = \cot\phi = 2\cot\theta,$$

$$\frac{dr}{r} = \frac{2\cos\theta d\theta}{\sin\theta} = \frac{2d\sin\theta}{\sin\theta},$$

$$\log r = 2\log\sin\theta + \text{const.}$$

$$r = c\sin^2\theta,$$

which is the polar equation to a line of force. Its form shews that all the lines of force are similar. The expression for the radius of curvature will be found to be

$$\rho = c\sin\theta\,\frac{(4-3\sin^2\theta)^{\frac{3}{2}}}{6-3\sin^2\theta},$$

shewing that ρ becomes indefinitely small as we approach the magnet.

The following is a convenient construction for drawing one of these curves.

Describe a semicircle with radius c. Let OA be any radius. Drop the perpendicular AB on the axis of the semicircle. Then if the angle which OA makes with the base of the semicircle is called θ, OB is $c\sin\theta$. Drop BP perpendicular on OA; then OP is $c\sin^2\theta$ and P is a point on the locus. See *Nature*, Feb. 19, 1880.

247. Let X, Y, Z be the components of attraction at a point (x, y, z) reckoned positive when they are in the directions of the positive axes, and let the theorem $\int N dS = 4\pi M$ be applied to a very small cube or right solid

having its centre at (x, y, z) and its edges parallel to the axes, their lengths being a, b, c. The average value of N over any one of the six faces is, to the first order of small quantities, the same as the value of N at the centre of this face. For the two faces which are perpendicular to the axis of x the co-ordinates of the centres are $x-\frac{a}{2}, y, z$; $x+\frac{a}{2}, y, z$ and the x components of attraction are $X-\frac{dX}{dx}\frac{a}{2}$, $X+\frac{dX}{dx}\frac{a}{2}$. The first of these is the value of N at the first face and the second is *minus* the value of N at the second face; hence the value of $\int N dS$ over this pair of opposite faces is $-\frac{dX}{dx}a$ multiplied by the area of either face. This area is bc, hence the product is $-\frac{dX}{dx}abc$; and treating the other pairs of faces in the same way we have

$$-\left(\frac{dX}{dx}+\frac{dY}{dy}+\frac{dZ}{dz}\right)abc$$

as the value of $\int N dS$ for the whole surface. This is to be equated to 4π multiplied by the included mass, that is to $4\pi abc\rho$, where ρ denotes the mean density of the solid, which is ultimately the density at the point (x, y, z). We have therefore

$$\frac{dX}{dx}+\frac{dY}{dy}+\frac{dZ}{dz}=-4\pi\rho \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

which may also be written

$$\frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}+\frac{d^2V}{dz^2}=-4\pi\rho \quad \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

ρ denoting the density at the point (x, y, z).

If the point is external to the attracting matter, ρ is zero and we have

$$\frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}+\frac{d^2V}{dz^2}=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (3),$$

an equation which is easily verified by differentiating the value of V due to the attraction of a single particle. For, taking the origin at the particle, we have

$$r^2 = x^2 + y^2 + z^2, \quad rdr = xdx + ydy + zdz,$$

$$\frac{dr}{dx} = \frac{x}{r}, \quad \frac{dr}{dy} = \frac{y}{r}, \quad \frac{dr}{dz} = \frac{z}{r},$$

$$\frac{d}{dx}\frac{1}{r} = -\frac{1}{r^2}\frac{dr}{dx} = -\frac{x}{r^3},$$

$$\frac{d^2}{dx^2}\frac{1}{r} = \frac{3x}{r^4}\frac{x}{r} - \frac{1}{r^3}.$$

Thus $$\frac{d^2}{dx^2}\left(\frac{1}{r}\right) + \frac{d^2}{dy^2}\left(\frac{1}{r}\right) + \frac{d^2}{dz^2}\left(\frac{1}{r}\right) = \frac{3(x^2+y^2+z^2)}{r^5} - \frac{3}{r^3} = 0,$$

and the result will still be zero when $\frac{m}{r}$ is put in the place of $\frac{1}{r}$. Since V is the sum of the potentials due to the several particles, $\frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} + \frac{d^2V}{dz^2}$ is the sum of a number of terms each of which is zero.

248. The following propositions will illustrate the subject of the present Chapter.

I. To find the attraction of a uniform lamina in the form of a regular polygon on a particle situated in a straight line drawn through the centre of the lamina at right angles to its plane.

Let n be the number of sides in the polygon, a the length of the perpendicular from the centre of the polygon on a side. Let axes of x and y be drawn through the centre of the polygon, the axis of x being perpendicular to a side. Let c be the distance of the particle from the lamina. The resultant attraction acts along the straight line which joins the particle with the centre of the lamina; and its value is

$$\mu \iint \frac{c\,dx\,dy}{(c^2 + x^2 + y^2)^{\frac{3}{2}}},$$

where μ is the product of the mass of the particle into the thickness and density of the lamina.

The integration must extend over the area of the polygon. To effect the integration it is convenient to transform to polar co-ordinates; thus we obtain

$$\mu c \iint \frac{r\,dr\,d\theta}{(c^2+r^2)^{\frac{3}{2}}}.$$

We must integrate with respect to r from $r=0$ to $r=a\sec\theta$, and then with respect to θ from $\theta=0$ to $\theta=\frac{\pi}{n}$; and multiply the result by $2n$.

Now $$\int \frac{r\,dr}{(c^2+r^2)^{\frac{3}{2}}} = -\frac{1}{(c^2+r^2)^{\frac{1}{2}}};$$

taking this between the limits we obtain

$$\frac{1}{c} - \frac{\cos\theta}{\sqrt{c^2\cos^2\theta + a^2}}.$$

Hence the required result is

$$2n\,\mu c \int_0^{\frac{\pi}{n}} \left\{\frac{1}{c} - \frac{\cos\theta}{\sqrt{c^2\cos^2\theta+a^2}}\right\} d\theta,$$

that is

$$2\mu\pi - 2n\mu c \int_0^{\frac{\pi}{n}} \frac{\cos\theta\,d\theta}{\sqrt{(a^2+c^2-c^2\sin^2\theta)}},$$

that is

$$2\mu\pi - 2n\mu \sin^{-1}\frac{c\sin\frac{\pi}{n}}{\sqrt{(a^2+c^2)}}.$$

II. To find the attraction of a uniform lamina in the form of a rectangle on a particle situated in a straight line drawn through the centre of the lamina at right angles to its plane.

Let $2a$ and $2b$ be the length and breadth of the rectangle, c the distance of the particle from the lamina. Proceeding as before we obtain the expression

$$\mu c \iint \frac{r\,dr\,d\theta}{(c^2+r^2)^{\frac{3}{2}}}.$$

We have to divide the integral into two parts. For one part we integrate with respect to r from $r=0$ to $r=a\sec\theta$, and then with respect to θ from $\theta=0$ to $\theta=\tan^{-1}\frac{b}{a}$. For the other part we integrate with respect to r from $r=0$ to $r=b\operatorname{cosec}\theta$, and then with respect to θ from $\theta=\tan^{-1}\frac{b}{a}$ to $\theta=\frac{\pi}{2}$. We multiply the result by 4.

$$\int_0^{a\sec\theta}\frac{rdr}{(c^2+r^2)^{\frac{3}{2}}}=\frac{1}{c}-\frac{\cos\theta}{\sqrt{(c^2\cos^2\theta+a^2)}}.$$

Integrate with respect to θ; thus we get

$$\frac{\theta}{c}-\frac{1}{c}\sin^{-1}\frac{c\sin\theta}{\sqrt{(a^2+c^2)}}.$$

$$\int_0^{b\operatorname{cosec}\theta}\frac{rdr}{(c^2+r^2)^{\frac{3}{2}}}=\frac{1}{c}-\frac{\sin\theta}{\sqrt{(c^2\sin^2\theta+b^2)}}.$$

Integrate with respect to θ; thus we get

$$\frac{\theta}{c}+\frac{1}{c}\sin^{-1}\frac{c\cos\theta}{\sqrt{(b^2+c^2)}}.$$

Hence the required result is

$$4\mu\left\{\frac{\pi}{2}-\sin^{-1}\frac{cb}{\sqrt{(a^2+b^2)}\sqrt{(a^2+c^2)}}-\sin^{-1}\frac{ca}{\sqrt{(a^2+b^2)}\sqrt{(b^2+c^2)}}\right\},$$

$$\text{or } 4\mu\left\{\cos^{-1}\frac{cb}{\sqrt{(a^2+b^2)}\sqrt{(a^2+c^2)}}-\sin^{-1}\frac{ca}{\sqrt{(a^2+b^2)}\sqrt{(b^2+c^2)}}\right\},$$

$$\text{or}\qquad 4\mu\sin^{-1}\frac{ab}{\sqrt{(a^2+c^2)}\sqrt{(b^2+c^2)}}.$$

III. Required the form of a homogeneous solid of revolution of given volume, which shall exercise the greatest attraction in a given direction on a given particle, the attraction varying as any inverse power of the distance.

Take the given particle as the origin, and the given direction as the straight line from which to measure angular

distance; let r, θ be the polar co-ordinates of any point in a fixed plane passing through the given direction. Then if the attraction vary inversely as the n^{th} power of the distance, the attraction of an element whose co-ordinates are r and θ may be denoted by $\frac{\mu}{r^n}$; and the resolved part of this attraction in the given direction will be $\frac{\mu}{r^n}\cos\theta$. Hence the equation

$$\frac{\mu}{r^n}\cos\theta = \text{constant}$$

represents a curve such that a given element placed at any point of it will exert the same attraction on the given particle along the given direction. Hence this equation will represent the curve which by revolving round the given direction will generate the required solid of greatest attraction, the constant being determined so as to give to the solid the prescribed volume. It is obvious that such is the case, because the surface we thus obtain separates space into two parts, and any element outside the surface exercises a less attraction along the given direction than it would if placed within the surface.

The history of this problem will be found by consulting the entry *Silvabelle's Problem* in the Index to the *History of the Mathematical Theories of Attraction*...

IV. Every element of the arc of a polar curve attracts with a force which varies inversely as the n^{th} power of the distance: determine the form of the curve when the resultant attraction of any arc on a particle at the pole bisects the angle between the radii vectores of the extremities of the arc.

Take any two small arcs AA', BB' having no part in common and subtending equal angles at the pole O, and let the angle AOB lie within the angle $A'OB'$. Draw OC bisecting one of these angles and therefore also bisecting the other. Since the attractions of the whole arc $A'B'$ and of the part AB are each in the direction OC, therefore the attraction of the remainder—that is the resultantattraction of the two arcs AA',

BB', acts along OC. Hence the attractions of AA' and BB' are equal. That is we have ultimately $\frac{\delta s}{r^n} = k\delta\theta$, where k is a constant, or

$$\frac{ds}{d\theta} = kr^n.$$

Therefore $$r^2 + \left(\frac{dr}{d\theta}\right)^2 = k^2 r^{2n};$$

this leads to either $\frac{dr}{d\theta} = 0$, or else

$$\left(\frac{d\theta}{dr}\right)^2 = \frac{1}{k^2 r^{2n} - r^2}.$$

The former supposition makes r constant, and so gives a circle. Taking the latter, and putting $\frac{1}{u}$ for r, we have

$$d\theta = \frac{u^{n-2}\, du}{\sqrt{(k^2 - u^{2n-2})}},$$

so that $$(n-1)\,\theta + C = \sin^{-1}\frac{u^{n-1}}{k},$$

where C is a constant.

Therefore $$\frac{1}{r^{n-1}} = k \sin\{(n-1)\,\theta + C\}.$$

If $n = 2$ we obtain

$$1 = kr \sin(\theta + C),$$

which is the equation to a straight line: see Art. 204.

If $n = 3$ we obtain

$$1 = kr^2 \sin(2\theta + C),$$

which is the equation to a rectangular hyperbola, the pole being at the centre.

EXAMPLES.

In the following Examples the ordinary law of attraction is to be assumed, unless the contrary be stated.

1. A solid is generated by the revolution of a sector of a circle about one of its bounding radii; find the attraction on a particle at the centre. *Result.* $\pi a\rho \sin^2 \beta$.

2. The rim of a hemispherical bowl consists of matter repelling with a force varying directly as the distance; shew that a particle will rest when placed *anywhere* on the concave surface.

3. A tube in the form of a parabola is placed with its axis vertical and vertex downwards; a heavy particle is placed in the tube, and a repulsive force acts along the ordinate upon the particle: find the law of force that it may sustain the particle in any position.

4. A portion of a cylinder of uniform density is bounded by a spherical surface, the radius of which is greater than that of the cylinder, and the centre coincides with the middle point of the base; find the attraction on a particle at this point.

Result. $2\pi\rho a - \frac{\pi\rho a^2}{b}$, where a is the radius of the cylinder and b the radius of the sphere.

5. Find the resultant attraction of a spherical segment on a particle at its vertex.

Result. $$2\pi h\rho \left\{1 - \frac{1}{3}\sqrt{\left(\frac{2h}{a}\right)}\right\},$$

where a is the radius of the sphere and h the height of the segment.

6. Find the resultant attraction of a spherical segment on a particle at the centre of its base.

Result. $$\frac{2\pi h\rho}{3(a-h)^2}\{3a^2 - 3ah + h^2 - (2a-h)^{\frac{3}{2}}h^{\frac{1}{2}}\}.$$

7. Find the locus of a point such that its resultant attraction on a fixed straight line may always pass through a fixed point in the straight line. *Result.* A sphere.

8. Find the attraction of a segment of a paraboloid of revolution, bounded by a plane perpendicular to its axis, on a particle at the focus.

Result. $4\pi\rho a \log \frac{x+a}{a}$, where x is the distance of the bounding plane from the vertex.

9. Round the circumference of a circle n equal centres of force are ranged symmetrically; each force is repulsive and varies inversely as the m^{th} power of the distance. A particle is placed in the plane of the circle very near its centre; shew that approximately the resultant force on it tends to the centre of the circle and varies as the distance of the particle from the centre, except when $m = 1$.

10. Eight centres of force, resident in the corners of a cube, attract, according to the same law and with the same absolute intensity, a particle placed very near the centre of the cube; shew that their resultant attraction passes through the centre of the cube, unless the law of force be that of the inverse square of the distance.

11. If the law of force in the preceding example be that of the inverse square of the distance, find the approximate value of the attraction on a particle placed very near the centre.

Result. Take the centre of the cube as origin and the axes parallel to the edges of the cube; then if x, y, z be the coordinates of the particle the attraction parallel to the axis of x is approximately

$$\frac{56x}{9\,(a\sqrt{3})^5}\,(3y^2+3z^2-2x^2)$$

towards the origin; $2a$ being the length of an edge.

12. The attraction of a uniform rod of indefinite length on an external particle varies as (distance)$^{-1}$ of the point from the

rod. Prove this, and supposing the asymptotes of an hyperbola to consist of such material, shew that a particle will be in equilibrium at any point of the hyperbola, and that the pressure on the curve at any point is proportional to the length of the tangent intercepted by the asymptotes.

13. Shew that the attraction of a uniform rod forming the perimeter of a rectilinear figure in which a circle can be inscribed vanishes at the centre of the circle.

14. Shew that the lines of force due to two infinitely long rods in the line ab produced both ways, one of which is attractive and the other repulsive, are the series of ellipses described from the extremities a and b as foci, and that the equipotential surfaces are generated by the revolution of the confocal hyperbolas. (*Cambridge and Dublin Mathematical Journal*, Vol. III. p. 94.)

15. The resultant attraction of a particle which attracts according to the inverse cube of the distance on a plane lamina is the same as on that part of the spherical shell described about the particle as centre and touching the plane of the lamina, which is cut off by straight lines from the centre to the edge of the lamina.

16. A particle attracted by two centres of force at A and B is placed in a fixed groove. Shew that the particle remains at rest at whatever point it is placed, provided that the form of the groove be such that

$$(AP - c)\,(BP - c') = cc',$$

where c, c' are constants dependent upon the absolute forces.

17. If a portion of a thin spherical shell, whose projections upon the three co-ordinate planes through the centre are A, B, C, attract a particle at the centre with a force varying as any function of the distance, shew that the particle will begin to move in the direction of the straight line whose equations are

$$\frac{x}{A} = \frac{y}{B} = \frac{z}{C}.$$

18. The particles of a thin hemispherical shell attract with a force $= \mu$ (distance), and those of a right conical shell repel

with a force $= \mu$ (distance). The rims of their bases coincide, and their vertices are turned in opposite directions; shew that a particle will rest in the common axis produced at a distance from the vertex of the sphere = length of the axis of the cone, the vertical angle of the cone being $2 \tan^{-1} \frac{4}{3}$.

19. Shew that if the attraction vary inversely as the distance, the integral of Nds over the perimeter of a closed plane figure, (N denoting the normal component of attraction directed inwards due to any superficial distribution of matter in the plane of the figure,) is equal to $2\pi M$, where M denotes so much of the matter as lies within the figure.

20. Shew that if the attraction vary inversely as the distance, an indefinitely thin plane ring attracts a particle in the plane of the ring beyond its outer circumference in the same manner as if the mass of the ring were collected at its centre, and that it exerts no force on a particle within the ring.

21. Prove that the work done by the mutual attraction of particles coming together from infinite distance to form a uniform sphere of mass M and radius R is $\frac{3}{5}\frac{M^2}{R}$.

22. Shew that the moon's attraction is greater at the nearest point of the earth than at the centre by $\frac{2mr}{R^3}$, m denoting the moon's mass, r the earth's radius, and R the moon's distance.

23. Understanding by "the moon's disturbing force" the resultant obtained by compounding the moon's attraction at any point on the earth's surface with a force equal parallel and opposite to the moon's attraction at the earth's centre, shew that the moon's disturbing force at a point at angular distance θ from the line of centres can be resolved into $M \sin \theta$ perpendicular to the line of centres and $2M \cos \theta$ parallel to the line of centres, M denoting $\frac{mr}{R^3}$. Shew also that the horizontal component of the disturbing force is $\frac{3}{2}M \sin 2\theta$.

CHAPTER XIV.

WORK, AND THE PRINCIPLE OF VIRTUAL VELOCITIES.

249. When the point of application of a force moves in a direction not at right angles to the force, the force is said to *work* or to *do work*. If the motion is in a straight line, through a distance s, and the force is constant in direction and of constant magnitude F, the amount of work done by the force is

$$Fs \cos \theta,$$

θ denoting the angle between the direction of the force and that of the motion. Since $s \cos \theta$ is the projection of the motion on the direction of the force, and $F \cos \theta$ is the component of the force along the line of motion, the work may be regarded at pleasure either as the product of the force by the effective component of the displacement or as the product of the displacement by the effective component of the force. It is positive or negative according as the angle between the force and the direction of motion is acute or obtuse.

When the motion is not restricted to a straight line, and the force is not restricted to be constant either in magnitude or direction, the work done is computed by applying the above definition to each successive small portion of the motion, and is therefore

$$\int F \cos \theta \, ds,$$

ds denoting an element of the path of the point of application, and θ the angle between F and ds.

If we reverse the sign of the *work done by* a force, the result is called the *work done against* the force.

250. Let dt be the elementary time in which the elementary distance ds is described, and let v be the velocity $\frac{ds}{dt}$; then since the work $Fds \cos\theta$ is done by the force in the time dt, the rate at which work is done per unit time is

$$Fv \cos\theta.$$

We shall call this the *work-rate* of the force. The work-rate of a given force, for a given velocity of its point of application, may therefore be defined either as the product of the force by the component velocity along its line of action, or as the product of the velocity by the tangential component of the force. By the work-rate of a number of forces (in a specified motion) is meant the algebraic sum of their separate work-rates.

251. The "Principle of Virtual Velocities" relates to the work-rate of the forces which keep a body or system of bodies in equilibrium.

In the more direct and natural application of the principle, the system must be supposed to be one which could be moved away from the position of equilibrium by even the smallest disturbing force. In any small displacement such as could be produced by a small disturbing force, some or all of the original forces do work either positive or negative. If the work done by one of these forces is regarded as a small quantity of the first order, the principle of virtual velocities asserts that the algebraic total of work done by all the forces of the system will either be zero or a small quantity of the second or some higher order.

Let the point of application of some one of the working forces be supposed to move uniformly through a small distance δs, and let δW be the joint work of all the forces during this motion; then the quotient $\delta W/\delta s$ will diminish indefinitely as δs diminishes, in other words the differential coefficient dW/ds will be zero. Since we suppose δs to be described

uniformly, δs is proportional to the time δt, and the work-rate $\frac{dW}{dt}$ will be zero.

The principle further asserts conversely that if a position be not one of equilibrium there will be some displacements for which the work-rate will not vanish.

252. With this preliminary explanation of what the Principle of Virtual Velocities is, we proceed to prove it. We shall first establish two elementary propositions respecting the work-rate of a force F whose point of application has the velocity v.

Proposition I. If the velocity v be resolved into any components v_1, v_2, &c., the work-rates of F for v_1, v_2, &c. will together be equal to the work-rate of F for v.

For, resolving along the direction of F, we have

$$v_1 \cos\theta_1 + v_2 \cos\theta_2 + \&c. = v,$$

therefore $$Fv_1 \cos\theta_1 + Fv_2 \cos\theta_2 + \&c. = Fv.$$

Proposition II. If the force F be resolved into any components F_1, F_2, &c., all having the same point of application as F, the work-rates of these components are together equal to the work-rate of F.

For, resolving along the direction of v, we have

$$F_1 \cos\theta_1 + F_2 \cos\theta_2 + \&c. = F\cos\theta,$$

therefore $$F_1 v \cos\theta_1 + F_2 v \cos\theta_2 + \&c. = Fv\cos\theta.$$

It follows that if two sets of forces acting at a point have the same resultant, they have the same work-rate, and that *if a set of forces acting at a point are in equilibrium, their work-rate is zero.*

Resolving F into rectangular components X, Y, Z, and v into rectangular components $\dot{x}$, $\dot{y}$, $\dot{z}$, the work-rate of X for $\dot{y}$ or $\dot{z}$ is zero, and the whole work-rate is $X\dot{x} + Y\dot{y} + Z\dot{z}$, a result which could also have been deduced from the formula $Fv\cos\theta$ by expressing $\cos\theta$ in terms of the direction cosines of F and v.

The element of work in moving through an elementary arc ds is $Xdx + Ydy + Zdz$, and the work in a finite path is $\int(Xdx + Ydy + Zdz)$.

253. Prop. III. If two equal and opposite forces applied at points A, B have AB as their common line of action, their joint work-rate is the product of either of them by the rate at which the distance AB changes.

Proof 1. Resolve the velocities of A and B along and perpendicular to AB. The perpendicular components do not affect the work-rates, and the joint work-rate will be the product of one of the forces by the difference of the component velocities along AB. But this difference is the rate at which the distance AB increases or diminishes.

As regards sign, if the directions of the forces are $\begin{cases}\text{outwards}\\ \text{inwards}\end{cases}$ the work rate is the product of either force by the rate at which the distance AB $\begin{cases}\text{increases}\\ \text{diminishes}\end{cases}$.

The following analytical investigation confirms this result:

Proof 2. Let $x_1y_1z_1$, $x_2y_2z_2$ be the co-ordinates of the two points of application. Then the work-rate is

$$X\dot{x}_2 + Y\dot{y}_2 + Z\dot{z}_2 - (X\dot{x}_1 + Y\dot{y}_1 + Z\dot{z}_1)$$
$$= X(\dot{x}_2 - \dot{x}_1) + Y(\dot{y}_2 - \dot{y}_1) + Z(\dot{z}_2 - \dot{z}_1).$$

But if l denote the distance between the points, and F the resultant of X, Y, Z, we have

$$X = F\frac{x_2 - x_1}{l}, \quad Y = F\frac{y_2 - y_1}{l}, \quad Z = F\frac{z_2 - z_1}{l};$$

therefore the work-rate is

$$\frac{F}{l}\{(x_2 - x_1)(\dot{x}_2 - \dot{x}_1) + (y_2 - y_1)(\dot{y}_2 - \dot{y}_1) + (z_2 - z_1)(\dot{z}_2 - \dot{z}_1)\}$$

$$= \frac{F}{l}\,\tfrac{1}{2}\frac{d}{dt}\{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\}$$

$$= \frac{F}{l}\,\tfrac{1}{2}\frac{d}{dt}l^2 = F\dot{l}.$$

As a particular case of special importance, we have the following proposition;—

Prop. IV. If two equal and opposite forces act along the same line at two points which are rigidly connected, their joint work-rate is zero in every motion.

254. From Props. II. and IV. we can draw the following inference;—

Prop. V. If two systems of forces applied to particles of a rigid body are statically equivalent, their work-rates are the same when the body is supposed to move.

For one of the two systems can be transformed into the other by the two following operations (see Chaps. II.—V.):

1. Replacing a force or forces acting at a point by an equivalent force or forces acting at the same point;

2. Introduction or removal of a pair of equal and opposite forces acting along the same line at two rigidly connected points;

and each of these operations leaves the work-rate unaffected.

By reversing the forces of one of the two systems, we obtain two systems which balance each other and have a joint work-rate of zero. Hence;—

Prop. VI. If forces applied to a rigid body are in equilibrium, their work-rate is zero in every motion of the body.

255. If any of the forces concerned change abruptly when the body moves, their work-rates (to satisfy Prop. VI.) must be computed by ignoring these changes and assigning to each force the magnitude and direction which it had when it equilibrated the rest. Thus, if we suppose a rigid body to move away from a smooth rigid surface whose reaction is one of the forces which keep it in equilibrium, we must multiply this reaction by the normal velocity of the point which was in contact with the surface, and include the product among the terms which compose the work-rate.

If we suppose the body to slide along a rough surface, we must reckon the work-rate of the force of friction as if this force retained its original magnitude and direction unchanged, ignoring the fact that the actual force of friction will be reversed in direction when the motion is reversed.

In order to avoid such strained interpretations of the Principle of Virtual Velocities as these, we must exclude all displacements that involve discontinuity in the forces of the system.

256. In any motion of sliding against a smooth surface, the work-rate of the reaction is zero, because the normal component of the velocity vanishes. The work-rate of the reaction is also zero in continuous rolling (for example in the rolling of a cylinder as distinguished from the rolling of a prism); for it can be shewn that the particles of the rolling body which are in contact with the surface rolled upon are for the instant at rest.

257. Prop. VII. If a system of particles connected by elastic threads without weight is in equilibrium under the action of external forces applied to some or all of the particles, the joint work-rate of the external and internal forces vanishes for every motion that the system can be supposed to take.

For the whole work-rate is the sum of the work-rates for the individual particles; and the work-rate for each particle is zero, because the particle is in equilibrium.

The work-rate of the external forces is therefore equal and opposite in sign to that of the internal forces, and does not vanish in a motion which stretches the threads.

On the other hand if we suppose inextensible threads, the internal work-rate is zero by Prop. IV., and the external work-rate is therefore also zero. We have thus another proof of Prop. VI., since a rigid body may be regarded as a collection of particles held at constant distances from each other.

In the same way it can be shewn that when any system of rigid bodies connected one to another is in equilibrium, the work-rate for all its forces together both external and internal is zero for every hypothetical displacement; and hence that when the work of the internal forces vanishes the work-rate of the external forces must also vanish. One case in which the-work rate of the internal forces vanishes, is that in which the connections consist of flexible and inextensible cords either knotted together or passing over smooth pullies.

258. Definition. A body or system of bodies is said to be *completely constrained*, or to have *only one degree of freedom* when its movements are so restricted that all its possible positions can be specified by the values of a single variable. A nut turning on a fixed screw is a good illustration. Each particle of such a body or system is restricted to a definite path; and if a single particle be prevented from moving along its path the whole body or system will be kept at rest. The single variable employed to specify the position of the system may be either linear or angular; let it be denoted by θ. Then two opposite motions from a given position are possible, one being specified by an increase and the other by a diminution of θ. It is to these two opposite motions, or opposite directions of motion, that the following proposition refers.

259. Prop. VIII. When a system with one degree of freedom is not in equilibrium, the direction in which it tends to move is that for which the work-rate of its forces is positive.

Take first the case of a single particle. Since it is not in equilibrium, the applied forces have a finite tangential component along the line in which the particle can move. The motion will be in the direction of this component, and the work-rate will therefore be positive.

Next take the case of a rigid body. Since it is not in equilibrium it will move, and since it has only one degree of freedom it can be prevented from moving by holding a single particle of it at rest. This can be done by applying to the particle a force of suitable magnitude in the opposite direction to that in which the particle tends to move. Now move the body in the direction in which the given forces tend to move it. The new equilibrating force will have a negative work-rate. But the work-rate of the given forces and the new force together must be zero, because they are in equilibrium. Therefore the work-rate of the given forces is positive.

In the case of a rigid body with more than one degree of freedom (that is, requiring more than one variable to specify

its possible positions), the forces applied to it, if not in equilibrium, will give it some definite initial motion. The work which they do in this motion will be the same as if the body were restricted to this motion alone. Hence the above conclusion is still applicable. The reasoning still holds if we substitute "system" for "rigid body"; hence *the direction in which a body or system tends to move is always a direction for which the work-rate of its forces is positive;* in other words *the forces of a system, when not in equilibrium, tend to do positive work.*

260. Prop. IX. If the work-rate of the forces applied to a rigid body is zero for every hypothetical motion through a certain position, that position is one of equilibrium.

For if the body, when in this position, is not in equilibrium, it will, when held at rest and then set free, begin to move in some definite way. Let frictionless constraints be imposed limiting the body to this particular motion. Since the body will then have only one degree of freedom, it can be held at rest by a single force applied to one particle in the opposite direction to that in which the particle tends to move. This force will have a negative work-rate in the supposed motion, and by hypothesis, the other forces have a zero work-rate; therefore the whole work-rate is negative although the forces are in equilibrium,—a result which we know to be impossible.

By extending this reasoning to a connected system of rigid bodies, we can establish the two following propositions;—

Prop. X. Let a system of rigid bodies be so connected that the internal forces between its parts can do no work when it is displaced; then if the work-rate of the forces applied to the system vanishes for every possible motion of the system through a given position, that position is one of equilibrium.

Prop. XI. Any connected system of rigid bodies is in equilibrium in a given position if the joint work-rate of the external and internal forces vanishes for every motion of the system through that position.

261. A body which is not rigid may be regarded as a collection of particles or rigid bodies between which mutual forces are exerted depending on their relative positions. Propositions VII. and XI. may thus be extended to elastic solids and even to liquids and gases, and the following general conclusion is established;—

It is necessary and sufficient for the equilibrium of any system of bodies, in a given position and subject to the action of given forces, that the work-rate of the forces shall vanish for every motion of the system through the given position. This is the *Principle of Virtual Velocities* in its most general form.

262. The following *reductio ad absurdum* affords a valuable confirmation of the foregoing results.

If possible let the work-rate of a system for a specified motion through a position of equilibrium be not zero.

Introduce frictionless constraints, permitting this motion only, and thus limiting the system to one degree of freedom. The displacements of the system can now be specified by the values of a single variable θ. Reckon as the positive direction for θ the direction of motion for which the work-rate is negative. Then if δW denote the work in the positive displacement $\delta\theta$, δW is negative and the limiting ratio $\frac{dW}{d\theta}$ is negative. Let $\frac{dW}{d\theta} = -A$. Then the work done is, to the first order of small quantities, $-A\delta\theta$, which is a negative quantity of the first order.

The forces can be divided into driving forces which do positive work and resisting forces which do negative work in the displacement considered. An increase of the first order in the driving forces will be sufficient to enable them to overpower the resisting forces and produce the displacement. The additional positive work thus introduced will be the product of a small quantity of the first order by $\delta\theta$, and will therefore be of the second order. Hence the whole work will still be correctly represented by $-A\delta\theta$ and will be negative, that is to say the driving forces as thus increased will do less work than is done against the resisting forces.

When the displacement $\delta\theta$ has been effected, let all the forces be shut off, and let the system be brought back to

its original position. This will not require the expenditure of any sensible amount of work. Then let the forces be allowed to act as before, and let these operations be repeated time after time. The system will in each complete cycle of operations give out more work than it takes in. Of the work given out, a portion sensibly equal to that taken in can be utilized for maintaining the driving forces, and the surplus will be available for industrial operations. We shall thus have a machine which drives itself and does useful work besides. Such an arrangement (which has often been imagined and attempted) is called a "perpetual motion". A wide induction from all departments of Physics has established its impossibility.

We now proceed to the distinction between stability and instability.

263. Let θ be the variable employed to specify the position of a system which has only one degree of freedom. Let δW be the work done by the forces of the system in the small displacement $\delta\theta$, and let δt be the time occupied. Then since $\frac{dW}{dt}$ is $\frac{dW}{d\theta}\frac{d\theta}{dt}$, the sign of $\frac{dW}{d\theta}$ is the same as the sign of the work-rate $\frac{dW}{dt}$ if θ is increasing, and is opposite to the sign of the work-rate if θ is diminishing. Hence, by Prop. VIII., the system tends to move so as to increase θ if $\frac{dW}{d\theta}$ is positive, and to move so as to diminish θ if $\frac{dW}{d\theta}$ is negative. When $\frac{dW}{d\theta}$ is zero, $\frac{dW}{dt}$ is also zero and the system is in equilibrium.

$\frac{dW}{d\theta}$ may be regarded as measuring the tendency of the forces of the system to increase θ, and if δW_1, δW_2,... are the amounts of work done in the displacement $\delta\theta$ by the different forces F_1, F_2,... of the system, then $\frac{dW_1}{d\theta}$, $\frac{dW_2}{d\theta}$,... are the tendencies of F_1, F_2,... to increase θ. Their sum is the whole tendency $\frac{dW}{d\theta}$.

264. Equilibrium is said to be stable or unstable, for a given small displacement from a position of equilibrium, according as the forces of the system tend to decrease or to increase this displacement. If it is stable for small displacements in all possible directions, it is said to be *completely stable*, or simply to be *stable*. If it is unstable for small displacements in all possible directions it is said to be *completely unstable*. If it is stable for some directions of displacement and unstable for others, it is to be regarded as on the whole unstable.

When the function W of which $\frac{dW}{d\theta}$ is the differential coefficient is a maximum, the equilibrium is stable; for if we start from a position very near that of equilibrium, the direction of motion for increasing W (that is for doing positive work) is towards the maximum.

On the other hand if W is a minimum, the direction of motion for increasing W will be that which carries the body further from the position of equilibrium.

We have accordingly the following criteria so far as displacements depending on θ are concerned.

$\frac{dW}{d\theta} = 0$ is the condition of equilibrium.

When this condition is fulfilled, the equilibrium is stable if $\frac{d^2W}{d\theta^2}$ is negative, and unstable if positive.

When both $\frac{dW}{d\theta}$ and $\frac{d^2W}{d\theta^2}$ vanish, then if $\frac{d^3W}{d\theta^3}$ is finite the equilibrium is stable for one of the two opposite directions of displacement and unstable for the other. It is therefore unstable on the whole.

When the first coefficient which does not vanish is $\frac{d^4W}{d\theta^4}$, the equilibrium is stable if this is negative, unstable if positive[1].

[1] In the above investigation stability is discussed from a purely statical point of view, no account being taken of the influence of inertia. To shew that the influence of inertia does not alter the criterion of complete stability, consider a body which is completely stable from the statical point of view. Let e denote the smallest quantity of work applied from without that will

265. We shall now investigate a few formulae for work-rates.

To calculate the work-rate of a couple for rotation in its own plane, we may replace it by an equivalent couple having one of its two forces F applied at the centre of rotation, and the other at the end of an arm drawn from this centre. The former does no work, since its point of application is at rest; the point of application of the latter moves with a velocity ωa, ω denoting the angular velocity and a the arm of the couple. As the direction of this motion coincides with the line of action of the force, the work-rate is $F\omega a$ or $-F\omega a$ according as the motion is with or against the force; that is, it is the product of the moment of the couple by the angular velocity, and is positive or negative according as the rotation is with or against the couple.

When a body rotates in a plane perpendicular to that of a couple which acts upon it, we may replace the couple by an equivalent one whose two forces are perpendicular to the plane of rotation and therefore perpendicular to the motions of the points at which they are applied. The work-rate of a couple is accordingly zero for rotation in any plane perpendicular to its own.

For any motion of translation (that is any motion in which all points of the body have equal and parallel velocities) the work-rate of a couple is obviously zero, since its two forces do equal and opposite work.

When the body rotates in a plane inclined at an angle ϕ to the plane of the couple, let G denote the moment of the couple. Then we may replace the given couple by two component couples one in and the other perpendicular to the plane of rotation. The latter may be neglected, and the moment of the former is $G\cos\phi$. The work-rate is accordingly $G\omega\cos\phi$.

266. To calculate the work-rate of any system of forces applied to a rigid body moving in any manner, we may

suffice to move the body into a position from which it will not tend to return when considered statically, and let k^2 be any quantity less than e. Then if the body move through the position of equilibrium with kinetic energy k^2, its subsequent movements can never carry it into a position in which its potential energy exceeds k^2 and hence can never carry it as far as the limits of statical stability.

select an arbitrary point O of the body, and specify the motion by specifying the velocity V of the point O in direction and magnitude, together with the position of the axis through O round which the body is turning, and the angular velocity ω round this axis. We can replace the forces by a single force R applied at O together with a couple G.

Let θ be the angle between the directions of R and V, and ϕ the angle between the axis of the couple G and the axis of the rotation ω, or, what is the same thing, the angle between their planes. Then the work-rate of R is $RV \cos\theta$, and the work-rate of G is $G\omega \cos\phi$. The whole work-rate is therefore

$$RV \cos\theta + G\omega \cos\phi.$$

267. Another expression, which has the advantage of putting in evidence the number of independent variables, can be obtained by employing three rectangular axes of reference through O. The velocity V can be resolved into 3 component velocities u, v, w along Ox, Oy, Oz, and the force R can be resolved into 3 component forces X, Y, Z in these directions. The work-rates of X, Y, Z are Xu, Yv, Zw; hence the work-rate of R is $Xu + Yv + Zw$.

The angular velocity ω can be resolved into 3 component angular velocities p, q, r round Ox, Oy, Oz, and the couple G can be resolved into 3 component couples L, M, N whose work-rates are Lp, Mq, Nr. The work-rate of G is therefore $Lp + Mq + Nr$, and the work-rate of the whole system of applied forces is

$$Xu + Yv + Zw + Lp + Mq + Nr.$$

268. The 6 quantities $uvwpqr$ which specify the motion and may be called the velocity-elements, are all independent, and so are the 6 quantities $XYZLMN$, which specify the statical effect of the forces and may called the force-elements. If the system is in equilibrium we know by Chapter v. that each of the 6 force-elements vanishes; the work-rate will therefore vanish.

Conversely if the work-rate

$$Xu + Yv + Zw + Lp + Mq + Nr$$

is to vanish for all values of the velocity-elements $uvwpqr$, we must have

$$X=0,\ Y=0,\ Z=0,\ L=0,\ M=0,\ N=0,$$

and therefore by Chapter v. the body will be in equilibrium. We have thus a confirmation of Propositions vi. and ix.

269. Let 5 different sets of forces be applied to a rigid body. Let the first set be specified by $X_1Y_1Z_1L_1M_1N_1$, the second set by $X_2Y_2Z_2L_2M_2N_2$, and so on, and let the motion of the body be specified by $uvwpqr$. The conditions of zero work-rates for the separate sets are

$$X_1u+Y_1v+Z_1w+L_1p+M_1q+N_1r=0,$$
$$X_2u+Y_2v+Z_2w+L_2p+M_2q+N_2r=0,$$
$$\&c. \qquad \&c. \qquad \&c.$$

It is always possible to find values of the 5 ratios of $uvwpqr$ which will satisfy these 5 equations, and there is generally only one solution.

Hence when 5 different sets of forces which can act on a rigid body are given, there is one definite motion possible in which no work will be done by any of the 5 sets. The body can be limited to this one motion by mounting it on a screw, and will then be in equilibrium under each of the 5 sets separately.

Each of the above 5 equations puts one degree of constraint on the body or in other words takes away one of its 6 degrees of freedom. Conversely when a rigid body has one degree of constraint, a definite set of forces can be found which will have a zero work-rate for all possible motions of the body.

In these statements we regard a set of forces as being defined by the 5 ratios of $XYZLMN$, and if there is to be only one solution of the 5 simultaneous equations the 5 sets must be really distinct, that is it must not be possible to express one set by sums and differences of the other sets.

This connection between zero work-rates and degrees of constraint was first pointed out by Sir Robert Ball. Reference may be made to *Ball's Theory of Screws*, or to a paper "On a new method in Statics and Kinematics" in the *Messenger of Mathematics*, for 1874.

270. The following proposition is often useful in applications of the principle of Virtual Velocities.

The work done by gravity upon a system of bodies in any displacement is equal to the product of their total weight by the vertical distance through which the centre of gravity of the system has descended.

Let $z_1, z_2 \ldots$ denote the initial and $z_1', z_2' \ldots$ the final distances of the particles below a fixed horizontal plane, $w_1, w_2 \ldots$ the weights of the particles. The works done by gravity upon the several particles are $w_1(z_1' - z_1)$, $w_2(z_2' - z_2) \ldots$; hence the whole work done by gravity upon the system is

$$\Sigma(wz') - \Sigma(wz).$$

But if $\bar{z}$ and $\bar{z}'$ denote the initial and final distances of the centre of gravity below the fixed plane, we have $\Sigma wz = \bar{z}\Sigma w$, $\Sigma wz' = \bar{z}'\Sigma w$, therefore the whole work is $(\bar{z}' - \bar{z})\,\Sigma w$.

Hence the work-rate for all the forces of gravity upon a system is the product of the weight of the system by the vertical downward velocity of its centre of gravity; and if $\dot{z}_1$, $\dot{z}_2 \ldots$ be the vertical downward velocities of the respective centres of gravity of the different portions of the system, and $w_1, w_2 \ldots$ the weights of these portions, the work-rate is $w_1\dot{z}_2 + w_2\dot{z}_1 + \&c.$

271. When there are no forces except gravity which can do work upon a system, the work-rate of gravity must vanish for every motion of the system through a position of equilibrium; hence when the system is passing through a position of equilibrium the centre of gravity must either be at rest or moving horizontally. Cases in which it is at rest are of rare occurrence. When it is moving horizontally, it must be at a point of its path at which the tangent is horizontal. If the locus of the possible positions of the centre of gravity is a curve, the equilibrium will be unstable at a summit, stable at the lowest point of a depression and unstable (on the whole) at a point of inflexion at which the tangent is horizontal. At a cusp, the centre of gravity will be stationary and there will be unstable equilibrium if the cusp is pointed upwards, stable if downwards.

When the locus of the possible positions of the centre of gravity is a surface, there will be unstable equilibrium at points where the tangent plane is horizontal and the surface completely convex upwards, stable equilibrium at points where the tangent plane is horizontal and the surface completely convex downwards, and unstable equilibrium (on the whole) at points where (the tangent plane being horizontal) there is convexity upwards in some sections and downwards in others (as at the central part of a saddle).

272. We shall illustrate the principle of virtual velocities by a few examples.

I. A beam rests on a smooth horizontal rail B and against a smooth wall at A, the rail being parallel to the wall and the beam being in a vertical plane perpendicular to the rail; to find the position of equilibrium and to determine whether it is stable or unstable.

Let b denote the distance of B from the wall, a the distance of the centre of gravity G of the beam from the end A, θ the inclination of the beam to the horizon.

The height of G above A is $a \sin\theta$ and the depth of A below B is $b \tan\theta$, hence the height of G above B is $a \sin\theta - b \tan\theta$, which we shall denote by u.

We have $$\frac{du}{d\theta} = a\cos\theta - b\sec^2\theta,$$

$$\frac{d^2u}{d^2\theta} = -a\sin\theta - 2b\sec^2\theta\tan\theta.$$

For equilibrium we have $\frac{du}{d\theta} = 0$, whence $\cos^3\theta = \frac{b}{a}$; and since both terms of the expression for $\frac{d^2u}{d\theta^2}$ are negative for all acute angles $\frac{d^2u}{d\theta^2}$ is negative and u is a maximum. The centre of gravity is therefore at a maximum height and the equilibrium is unstable.

II. A jointed regular hexagon composed of uniform rods has its highest and lowest sides horizontal, their middle points being connected by a string and the highest side being supported. Required the tension T of the string, the weight of each side of the hexagon being w.

Suppose the string to shorten by an amount s. The work done by the tension will be Ts, and must be equal to the work done against gravity, which is ws for the lowest side, and $4w\frac{s}{2}$ for the other four moveable sides, since their common centre of gravity is half-way between the highest and lowest sides.

We have therefore

$$T = 3w.$$

III. A string of given length has one end fixed at a point in the line of intersection of two vertical planes at right angles to each other, and at the other end carries a heavy particle which is repelled from these planes by forces of which one is constant and the other varies as the distance from the plane; find the positions of equilibrium.

Take the vertical plane from which the particle is repelled by a constant force as the plane of (x, z), and the other vertical plane as the plane of (y, z); take the point to which the end of the string is fixed as the origin, and let the axis of z be vertically downwards. Let x, y, z denote the co-ordinates of the particle in a position of equilibrium, and l the length of the string. Let W be the weight of the particle, F the constant repulsive force, μx the force which varies as the distance of the particle from the plane of (y, z).

Conceive the particle displaced with component velocities $\dot{x}, \dot{y}, \dot{z}$. Then by the principle of virtual velocities

$$\mu x\dot{x} + F\dot{y} + W\dot{z} = 0 \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Also
$$x^2 + y^2 + z^2 = l^2 \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

therefore
$$x\dot{x} + y\dot{y} + z\dot{z} = 0 \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Eliminating $\dot{z}$ between (1) and (3) we have

$$\dot{x}(Wx - \mu xz) = \dot{y}(Fz - Wy),$$

and since this is to be true for all values of the ratio of $\dot{x}$ to $\dot{y}$ we must have

$$Wx - \mu xz = 0, \qquad Fz - Wy = 0.$$

From the first of these equations we obtain either $z = \dfrac{W}{\mu}$ or else $x = 0$. If we take the former solution, we obtain $y = \dfrac{F}{\mu}$, and then x is known from (2); thus one position of equilibrium is determined. If we take the solution $x = 0$, then y and z must be found from the equations

$$Fz - Wy = 0, \qquad y^2 + z^2 = l^2;$$

thus another position of equilibrium is determined.

IV. A uniform heavy beam is placed with its ends in contact with a fixed smooth vertical curve in the form of an ellipse with its directrices horizontal: determine the position of stable equilibrium, the length of the beam being supposed not less than the latus rectum.

Let P and Q denote the extremities of the beam; let PM and QN be perpendiculars on the lower directrix, and S the focus corresponding to this directrix. Then the height of the centre of gravity of the beam above this directrix is $\frac{1}{2}(PM + QN)$; for stable equilibrium this height should be a minimum. If e be the excentricity of the ellipse we have

$$PM + QN = \frac{1}{e}(SP + SQ);$$

and therefore $SP + SQ$ must be a minimum. But $SP + SQ$ is always greater than PQ, except when S is in the straight line PQ. Therefore the position of stable equilibrium is that in which PQ passes through the focus.

Since the beam is in equilibrium under the action of its own weight and the normal resistances of the curve, it follows that the straight line which joins the point of intersection of normals at the ends of a focal chord of an ellipse with the middle point of the chord is parallel to the major axis: this result may be verified geometrically.

V. The principle of Art. 270 may be applied to a liquid, which may be regarded as a collection of indefinitely small smooth heavy particles.

Suppose a set of rectangles, all of the same length, but with any breadths. Let them be connected along their lengths by smooth hinges, so as to form a hollow prism without ends; and place the system vertically on a smooth horizontal plane. Let some liquid be poured into the vessel thus formed. In the position of stable equilibrium the centre of gravity of the liquid will be at a minimum height above the horizontal plane; and therefore the area of a horizontal section of the prism will then have a maximum value. Hence if A, B, C, D be any four consecutive corners of the section, the area $ABCD$ must be a maximum for a given value of AD. But (*Diff. Cal.* Art. 240) when the four sides of a quadrilateral are given, the area is greatest when the quadrilateral can be inscribed in a circle. Hence in the position of equilibrium the horizontal section of the prism must form a polygon which can be inscribed in a circle. See also Prop. II. at the end of Chap. IV.

VI. To find the curve of given length joining two fixed points which has its centre of gravity at the greatest distance from the straight line joining the points.

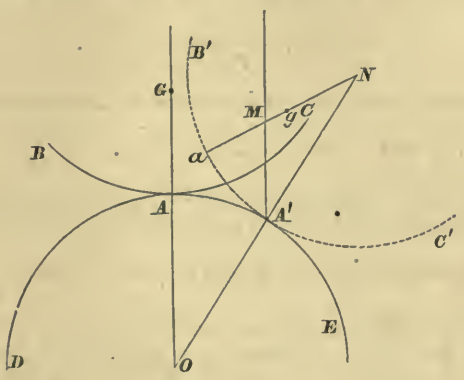
Let the joining straight line be placed in a horizontal position, and let a uniform chain of the given length be attached by its ends to the two points. The chain will take a position of stable equilibrium under the action of gravity, and its centre of gravity will then be at the maximum depth below the horizontal line. Hence the required curve is the common catenary.

273. The following is a simple example of distinguishing the nature of equilibrium.

A heavy body rests on a fixed body; to determine the nature of the equilibrium, the surfaces being supposed rough.

Let BAC be a vertical section of the upper body made by a plane through its centre of gravity G, and DAE the section

of the lower body made by the same plane. We suppose these sections both circular; let r be the radius of the upper



section and R that of the lower. Let the upper body be displaced into the position $B'A'C'$, and suppose a that point in the upper body which was originally at A; at A' the new point of contact draw the common normal $OA'N$, meeting at O the radius AO of the lower surface, and at N the radius aN of the upper surface. Draw a vertical line through A' meeting aN at M; let g be the new position of the centre of gravity of the upper body. If we suppose the surface rough enough to prevent all sliding, the upper body will turn round A', and the equilibrium will be unstable if g falls further from a than M, and *stable* if g be between M and a.

Let $$AOA' = \theta, \quad aNA' = \phi.$$

Since we suppose the upper body displaced by *rolling* on the lower, we have

$$\text{arc } AA' = \text{arc } aA';$$

therefore $$R\theta = r\phi.$$

Also $$\frac{MN}{NA'} = \frac{\sin\theta}{\sin(\theta+\phi)} = \frac{\sin\theta}{\sin\left(1+\frac{R}{r}\right)\theta}.$$

$$= \frac{1}{1+\frac{R}{r}} \text{ ultimately;}$$

therefore $$MN = \frac{r^2}{r+R},$$

and $$aM = r - \frac{r^2}{r+R} = \frac{Rr}{R+r}.$$

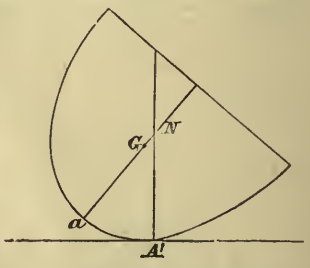
Hence, the equilibrium is stable or unstable according as ag, or AG, is less or greater than $\frac{Rr}{R+r}$.

If the lower surface be *concave* instead of *convex*, it may be shewn in the same way that the equilibrium is stable or unstable according as AG is less or greater than $\frac{Rr}{R-r}$.

The results of this Article will hold when the sections BAC and DAE are not circles; r and R will then stand for the radii of curvature of the upper and lower sections at the point A. If the lower surface is plane, R is infinite, and for stable equilibrium AG must be less than r.

274. If $AG = \frac{Rr}{R+r}$ in the first case, or $= \frac{Rr}{R-r}$ in the second case, the equilibrium is neutral to a first approximation, and a further investigation will have to be made to determine whether it is really stable or unstable. Suppose, for example, that the heavy body is a portion of a paraboloid resting in equilibrium with its vertex in contact with a horizontal plane.

The equilibrium is neutral to a first approximation when the centre of gravity G coincides with the centre of curvature of the generating parabola at the vertex. Now, if different points be taken in a parabola, the further the assumed point is from the vertex, the further is the point of intersection of the normal and the axis from the vertex. Hence, the normal $A'N$ in the figure meets the axis of the parabola further from a than G is, and the equilibrium is *stable*.



It is easy to shew generally, that if a portion of a solid of revolution rest in neutral equilibrium with its vertex on a horizontal plane, the equilibrium is really *stable* or *unstable*, according as the radius of curvature of the generating curve has a *minimum* or *maximum* value at the vertex.

275. The results of Art. 273, when the sections BAC and DAE are circles, may also be obtained by using the theorem of Art. 271.

Let z denote the height of the centre of gravity g above the horizontal line through O, and let $Ng = c$; then

$$z = (R+r)\cos\theta - c\cos(\theta+\phi)$$

$$= (R+r)\cos\theta - c\cos\left(1+\frac{R}{r}\right)\theta.$$

Expand the cosines in powers of the angles; thus

$$z = R+r-c+\left\{c\left(1+\frac{R}{r}\right)^2-(R+r)\right\}\frac{\theta^2}{2}$$

$$-\left\{c\left(1+\frac{R}{r}\right)^4-(R+r)\right\}\frac{\theta^4}{\lfloor 4}+\ldots$$

Suppose the coefficient of θ^2 not to be zero; then when θ is indefinitely small z is greater or less than $R+r-c$, according as the coefficient of θ^2 is positive or negative; in the former case $R+r-c$ is a *minimum* value of z, and in the latter case it is a *maximum* value. Therefore the equilibrium is stable if c be greater than $\frac{r^2}{R+r}$, and unstable if c be less than $\frac{r^2}{R+r}$.

Suppose however that $c = \frac{r^2}{R+r}$, then the coefficient of θ^2 is zero, and the equilibrium is neutral to a first approximation. We must now examine the coefficient of θ^4 in the value of z; this coefficient is

$$-\frac{1}{\lfloor 4}\left\{c\left(1+\frac{R}{r}\right)^4-(R+r)\right\},$$

that is,

$$-\frac{1}{\lfloor 4}\left\{\frac{(R+r)^5}{r^4}-(R+r)\right\},$$

that is, $$-\frac{R(R+r)(R+2r)}{r^2\lfloor 4}\,;$$

since this is a *negative* quantity it follows that $R+r-c$ is a *maximum* value of z and the equilibrium is really *unstable.*

276. The following problem will furnish an instructive example. A frame formed of four uniform rods of the length a connected by smooth hinges, is hung over two smooth pegs in the same horizontal line at a distance $\frac{a}{\sqrt{2}}$, the two pegs being in contact with different rods; shew that the frame is in equilibrium when each angle is 90^0, and determine whether the equilibrium is stable or unstable.

Denote the pegs by A and B; suppose the beam in contact with A to make an angle θ with the horizon, and the beam in contact with B to make an angle ϕ with the horizon; let u denote the depth of the centre of gravity of the system below AB. Then it may be shewn that

$$u=\frac{a}{2}(\sin\theta+\sin\phi)-\frac{c\sin\theta\sin\phi}{\sin(\theta+\phi)}\,,$$

where $$c=\frac{a}{\sqrt{2}}\,.$$

Thus u is a function of the two independent variables θ and ϕ, and in order that u may have a maximum or minimum value θ and ϕ must be taken so as to satisfy $\frac{du}{d\theta}=0$ and $\frac{du}{d\phi}=0$. It will be found on trial that $\theta=\frac{\pi}{4}$ and $\phi=\frac{\pi}{4}$ are suitable values. But it will be found that with these values for θ and ϕ we get

$$\frac{d^2u}{d\theta^2}=-\frac{c}{2},\quad \frac{d^2u}{d\theta\,d\phi}=-c,\quad \frac{d^2u}{d\phi^2}=-\frac{c}{2}\,;$$

so that $\left(\frac{d^2u}{d\theta\,d\phi}\right)^2-\frac{d^2u}{d\theta^2}\frac{du^2}{d\phi^2}$ is *positive* and u is *neither a maximum nor a minimum* when $\theta=\frac{\pi}{4}$ and $\phi=\frac{\pi}{4}$. All the

foregoing is a simple example of the Differential Calculus; we proceed to apply it to the Mechanical Problem in question.

Let δu denote the change in u consequent upon changing the value of θ from $\frac{\pi}{4}$ to $\frac{\pi}{4}+\delta\theta$, and the value of ϕ from $\frac{\pi}{4}$ to $\frac{\pi}{4}+\delta\phi$; then it follows from the preceding investigations that

$$\delta u = -\frac{c}{4}\{(\delta\theta)^2 + 4\delta\theta\delta\phi + (\delta\phi)^2\} + \&c.,$$

where under the &c. are included terms in $\delta\theta$ and $\delta\phi$ of a higher order than the second. Now although u is neither a maximum nor a minimum when θ and ϕ are each $\frac{\pi}{4}$, yet there is equilibrium then because δu is then zero so far as terms of the first order in $\delta\theta$ and $\delta\phi$. (See Arts. 270, 271.) But as u is *neither* a maximum *nor* a minimum the equilibrium cannot be stated to be either stable or unstable *universally*; it is in fact *stable* with respect to *some* displacements and *unstable* with respect to *other* displacements. If, for example, we consider only such displacements as make $\delta\theta = \delta\phi$, then δu is certainly *negative* when $\delta\theta$ and $\delta\phi$ are taken small enough; thus the centre of gravity is *raised* by the displacement and so the equilibrium is *stable*. If, again, we consider only such displacements as make $\delta\theta = -\delta\phi$, then δu is certainly *positive* when $\delta\theta$ and $\delta\phi$ are taken small enough; thus the centre of gravity is *depressed* by the displacement and so the equilibrium is *unstable*.

EXAMPLES.

1. A cone whose semi-vertical angle is $\tan^{-1}\frac{1}{\sqrt{2}}$ is enclosed in the circumscribing spherical surface; shew that it will rest in any position.

2. A heavy uniform rod of length a moves in a vertical plane about a hinge at one extremity. A string fastened

to the other, passes over a pully in a vertical line above the hinge, and is attached to a weight equal to half that of the rod, which rests on a curve. The length of the string and the height of the pully above the hinge are each equal to the length of the rod, and the system is in equilibrium in all positions. Shew that the equation to the curve is

$$r = 4a \sin^2 \tfrac{1}{2}\theta,$$

the pully being the origin and the prime radius being vertical.

3. Two rods each of length $2a$ have their ends united at an angle α, and are placed in a vertical plane on a sphere of radius r. Prove that the equilibrium is stable or unstable according as $\sin \alpha$ is $>$ or $< \dfrac{2r}{a}$.

4. An oblong cone rests with its smaller end on a horizontal table. Is the equilibrium stable or unstable?

5. A cylinder rests with the centre of its base in contact with the highest point of a fixed sphere, and four times the altitude of the cylinder is equal to a great circle of the sphere; supposing the surfaces in contact to be rough enough to prevent sliding, shew that the cylinder may be made to rock through an angle of 90^0, but not more, without falling off the sphere.

6. A very small bar of matter is moveable about one extremity which is fixed halfway between two centres of force attracting inversely as the square of the distance; if l be the length of the bar, and $2a$ the distance between the centres of force, prove that there will be two positions of equilibrium for the bar, or four, according as the ratio of the absolute intensity of the more powerful force to that of the less powerful, is, or is not, greater than $\dfrac{a+2l}{a-2l}$; and distinguish between the stable and unstable positions.

7. Two particles connected by a string support each other on the arc of a verticle circle; shew that the centre of gravity is in the vertical through the centre of the circle. What is the nature of the equilibrium?

8. A sphere of radius a, loaded so that the centre of gravity may be at a given distance b from the centre of figure, is placed on a rough plane inclined at an angle α to the horizon. Shew that there will be two positions of equilibrium, one stable and the other unstable, in which the distances of the point of contact from the centre of gravity are respectively,

$$a\cos\alpha - \sqrt{(b^2 - a^2\sin^2\alpha)},$$

and

$$a\cos\alpha + \sqrt{(b^2 - a^2\sin^2\alpha)}.$$

Hence, find the greatest inclination of the plane which will allow the sphere to rest. Is the equilibrium stable or unstable in this limiting case?

9. A sphere of radius r rests on a concave sphere of radius R; if the sphere be loaded so that the height of its centre of gravity from the point of contact be $\frac{3}{2}r$, find R so that the equilibrium may be neutral. *Result.* $R = 3r$.

10. A heavy cone rests with the centre of its base on the vertex of a fixed paraboloid of revolution; shew that the equilibrium will be neutral if the height of the cone be equal to twice the latus rectum of the generating parabola. Shew that the equilibrium is really stable.

11. A heavy particle attached to one extremity of an elastic string is placed upon a smooth curve, the string lying upon the curve and its other extremity being fixed to a point in the curve; find the curve when the particle rests in all positions.

Result. A cycloid.

12. A uniform square board is capable of motion in a vertical plane about a hinge at one of its angular points; a string attached to one of the nearest angular points, and passing over a pully vertically above the hinge at a distance from it equal to the side of the square, supports a weight whose ratio to the weight of the board is 1 to $\sqrt{2}$. Find the positions of equilibrium and determine whether they are respectively stable or unstable.

13. Two small smooth rings of equal weight slide on a fixed elliptical wire of which the major axis is vertical, and are connected by a string passing over a smooth peg at the upper focus; prove that the rings will rest in whatever position they may be placed.

14. A small heavy ring slides on a smooth wire in the form of a curve whose plane is vertical, and is connected by a string passing over a fixed pully in the plane of the curve with another weight which hangs freely; find the form of the curve that the ring may be in equilibrium in any position.

Result. A conic section having its focus at the pully.

15. If an elliptic board be placed, so that its plane is vertical, on two pegs which are in the same horizontal plane, there will be equilibrium if these pegs be at the extremities of a pair of conjugate diameters. What are the limits which the distance between the pegs must not exceed or fall short of, in order that this position of equilibrium may be possible? Shew that the equilibrium is *unstable*.

16. A solid of revolution, whose centre of gravity coincides with the centre of curvature at the vertex, rests on a rough horizontal plane. Shew that the equilibrium is stable or unstable according as the value of $3\left(\frac{d^2y}{dx^2}\right)^3 - \frac{d^4y}{dx^4}$, when x and y vanish, is positive or negative, x and y being co-ordinates of the generating curve, measured along the tangent and normal at the vertex.

17. If a plane pass through one extremity A of the base of a cylinder and be inclined at an angle of 45° to the axis, the piece so cut off will rest in neutral equilibrium, if placed with its circular end on the vertex of a paraboloid whose latus rectum is five-eights of the diameter of the base, the point of contact being also at this same distance from A.

18. A piece of string is fastened at its extremities to two fixed points; determine from mechanical considerations the form which must be assumed by the string in order that the surface generated by its revolution about the straight line joining the fixed points may be the greatest possible.

MISCELLANEOUS EXAMPLES.

1. A uniform wire is bent into the form of three sides AB, BC, CD of an equilateral polygon; and its centre of gravity is at the intersection of AC and BD. Shew that the polygon must be a regular hexagon.

2. Three forces act along three straight lines which may be considered as generating lines in the same system of a hyperboloid of one sheet; shew that if the forces admit of a single resultant, it must act along another generating line of the same system.

3. A gate moves freely about a vertical axis, along which it also slides; while a point in the plane of the gate, and rigidly connected with it, rests on a given rough inclined plane; find the limiting position of equilibrium.

4. Suppose straight lines to be drawn from one of the centres of the four circles that touch the sides or the sides produced of a given triangle to the other three centres, and let these straight lines represent three forces in magnitude and direction; then the straight line joining the first centre with the centre of the circle circumscribing the triangle will represent in magnitude and direction one-fourth of the resultant.

5. A particle rests in equilibrium in a fine groove in the form of a helix, the axis of which is inclined to the horizon at a given angle α. Find the distance of the particle from a vertical plane passing through the axis. Also find the greatest value of α for a given helix in order that there may be a position of equilibrium of the particle.

6. A quadrilateral figure possesses the following property; any point being taken and four triangles formed by joining this point with the angular points of the figure, the centres of gravity of these triangles lie in the circumference of a circle; prove that the diagonals of the quadrilateral are at right angles to each other.

7. A square board is supported in a horizontal position by three vertical strings; if one of them be attached to a corner, where must the others be attached in order that the weight which can be placed on any part of the board without overturning it may be the greatest possible?

8. A triangular plate hangs by three parallel threads attached at the corners, and supports a heavy particle. Prove that if the threads are of equal strength, a heavier particle may be supported at the centre of gravity than at any other point of the disc.

9. ABC is a triangle; D, E, F are the middle points of the sides opposite to A, B, C respectively; P is any point; PD, PE, PF are divided in a given ratio at A', B', C' respectively: shew by the theory of the centre of gravity that AA', BB', and CC' meet at a point.

10. A right cone is cut obliquely and then placed with its section on a horizontal plane; prove that when the angle of the cone is less than $\sin^{-1}\frac{1}{4}$, there will be two sections for which the equilibrium is neutral, and for intermediate sections the cone will fall over.

11. A right cylinder on an elliptic base (the semiaxes of which are a and b) rests with its axis horizontal between two smooth inclined planes inclined at right angles to each other; determine the positions of equilibrium, (1) when the inclination of one of the planes is greater than $\tan^{-1}\frac{a}{b}$, (2) when the inclination of both planes is less than $\tan^{-1}\frac{a}{b}$.

12. A pack of cards is laid on a table; each projects in the direction of the length of the pack beyond the one below it; if each projects as far as possible, prove that the distances between the extremities of the successive cards will form an harmonic progression.

13. Find the least excentricity of an ellipse in order that it may be capable of resting in equilibrium on a perfectly rough inclined plane.

Result. $e^2 = \dfrac{2\sin\alpha}{1+\sin\alpha}$.

14. Two mutually repelling particles are placed in a parabolic groove, and connected by a thread which passes through a small ring at the focus; shew that if the particles be at rest, either their abscissæ are equal, or the two parts of the thread form one straight line.

15. Each element of a parabolic arc bounded by the vertex and the latus rectum is acted on by a force in the normal proportional to the distance of the element from the axis of the parabola. Shew that the equation to the straight line in which the resultant acts is

$$15y + 10x = 26a.$$

16. Each element of the arc of an elliptic quadrant is acted on by a force in the normal proportional to the ordinate of that point. Shew that the equation to the straight line in which the resultant acts is

$$6by - 3\pi ax + 4a^2 - 4b^2 = 0.$$

17. A smooth body in the form of a sphere is divided into hemispheres and placed with the plane of division vertical upon a smooth horizontal plane; a string loaded at its extremities with two equal weights hangs upon the sphere, passing over its highest point and cutting the plane of division at right angles; find the least weight which will preserve the equilibrium.

18. The locus of the centre of gravity of segments of equal area A in an ellipse is a similiar concentric ellipse whose minor axis is

$$\frac{4ab^2}{3A}\sin^3\frac{\phi}{2}, \text{ where } A = \frac{ab}{2}(\phi - \sin\phi).$$

19. The foci of a rough oblongum attract directly as the distance; if a particle without weight be placed on the oblongum, find within what limits it must be placed so as to be in equilibrium. Shew that if the coefficient of friction be greater than $\frac{e^2}{2\sqrt{(1-e^2)}}$, where e is the excentricity, the particle will rest anywhere on the surface.

20. A circular disc of mass m' and radius c rests in contact with two equal uniform straight rods AB, AC, which are joined at A by a smooth hinge, and which atttract each other and the disc with a force varying as the distance; also the

disc attracts the rods similarly. Shew that there is equilibrium if

$$m'c\,(2c\cos\alpha - a\sin\alpha) = ma^2\sin^4\alpha\cos\alpha,$$

where m is the mass of each rod, a the length of each rod, and 2α their inclination to each other.

21. A square picture hangs in a vertical plane by a string, which passing over a smooth nail has its ends fastened to two points symmetrically situated in one side of the frame. Determine the positions of equilibrium, and whether they are stable or unstable.

Results. Let l be the length of the string, c the distance of the two points to which the ends of the string are fastened, h the length of a side of the square; then if lh be greater than $c\sqrt{(c^2+h^2)}$ there is only one position of equilibrium, namely, the ordinary position, and the equilibrium is *stable;* if lh be less than $c\sqrt{(c^2+h^2)}$ there are two oblique positions of *stable* equilibrium, besides the ordinary position of equilibrium, which is *stable* with respect to some displacements and *unstable* with respect to other displacements.

22. A flexible thread is placed in a tube of any form and is acted on by any forces. The diameter of the tube is equal to that of the thread and is infinitesimal. Determine the position of equilibrium.

23. Two equal particles are connected by two given strings without weight, which are placed like a necklace on a smooth cone with its axis vertical and vertex upwards; find the tensions of the strings.

24. A triangle of area A revolves through an angle ϕ about an axis in its own plane taken parallel to one side; shew that the least amount of surface generated is

$$A\,.\,\phi\,.\,\frac{(a+b+c)^2-2a^2}{2(b+c)a},$$

where a is the greatest side.

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